Combinatorial Optimization

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Abstract

These notes are from a course in Combinatorial Optimization, as offered in Fall 2015 at the University of Illinois at Urbana-Champaign. The professor for the course was Karthekeyan (Karthik) Chandrasekaran.

Information about the course can be found at the course web page, `http://karthik.ise.illinois.edu/courses/comb-opt/comb-opt.html`.

Lecture 1 is (likely) irrecoverably missing. Lecture 21 might one day be properly typed up.

These notes were transcribed by Patrick Lin and attempt to follow the lectures as faithfully as was possible, but there are likely many errors and inconsistencies.

Revision 1 Feb 2018 13:44.

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Lecture 1, August 25, 2015

Missing :(

Lecture 2, August 27, 2015

1 Matchings [Bipartite]

We consider the problem of Maximum Bipartite Matching.
Theorem 1.1 (König). *The size of the maximum bipartite matching is equal to the size of the minimum vertex cover.*

We will design an algorithm solving both simultaneously.

Recall the idea of an M-augmenting path: [**TODO Fill in**]
Theorem 1.2. *A matching M is of maximum cardinality iff there exists no M-augmenting path.*

Proof. M is of maximum cardinality $\implies$ no there is no M-augmenting path: If we had an M-augmenting path then we can get a bigger matching.

There is no M-augmenting path $\implies$ M is of maximum cardinality: Suppose M is not maximal, then there exists a matching M^*, $|M^*| \geqslant |M|$. Each component of $M \triangle M^*$ is an $M^{(*)}$-alternating cycle or an $M^{(*)}$-alternating path. So all cycles should be of even length, and so there exists a path with more M^* edges than M edges. This is an M-augmenting path. $\square$

ALGORITHM

Start with any matching M
repeat
  Find an M-augmenting path P
  if $\nexists P$
    Stop and **return** M
  else
    Augment M along P

Note that this works for all graphs, not just bipartite.

But: How do we find an M-augmenting path? For bipartite graphs, P should start in an exposed vertex in A and end at one in B. So we can just perform depth-first search where non-matching edges are from A to B, and matching edges are from B to A (call this directed graph D).
Claim 1.3. *There exists an M-augmenting path iff there exists a directed path in D from an exposed vertex of A to an exposed vertex in B.*

Proof. Exercise. $\square$

This implies that an augmenting path can be found in $O(|E|)$ time, and the overall runtime is $O(|V||E|)$.

1.1 Minimum Vertex Cover

Let R be the set of vertices reachable by directed paths in D from an exposed vertex in A.

Claim 1.4. *Suppose the algorithm terminates with no M-augmenting paths. Then $S = (A - R) \cup (B \cap R)$ is a vertex cover with $|S| = |M|$. In particular, $|S|$ is minimal.*

Proof. Assume S is not a vertex cover. Then there exists an edge $e = (a, b)$ with $a \in A \cap R$, $b \in B - R$. If $e \in M$ then $b \in B - R \implies a \notin R$, a contradiction. If $e \notin M$ then $a \in A \cap R \implies b \in R$, also a contradiction. So S is a vertex cover.

Next, we show that $|S| \leqslant |M|$. No vertex in $A - R$ is exposed by M by definition of reachability. If a vertex in $B - R$ is unmatched, then we have an M-augmenting path.

Claim 1.5. *These vertices are not matched to each other.*

> *Proof.* Suppose they are. M paths are oriented $B \rightsquigarrow A$, but then vertices in A are reachable. Contradiction. ■

So we have a pairing between vertices in S and matchings in M. Hence $|S| \leqslant |M|$. □

Theorem 1.6 (Hall)**.** *Let $G = (A \cup B, E)$ be bipartite. G has a matching of size $|A|$ iff $|N(S)| \geqslant |S|$ $\forall S \subseteq A$.*

Proof. Exercise (use König's Theorem). □

2 Minimum Cost Perfect Matching

Given: a complete bipartite graph with $|A| = |B| = n$ and $c_{ij} \in \mathbb{R} \cup \{\infty\}$, $i \in A, j \in B$.

Goal: Find a perfect matching M' with $\min_{ij \in M} c_{ij}$.

The following is an Integer Programming formulation for the problem, where $x_{ij} = 1 \iff ij \in M$:

$$\begin{aligned} \min \sum c_{ij} x_{ij} & \\ \sum_{j \in B} x_{ij} = 1 & \qquad \forall i \in A \\ \sum_{i \in A} x_{ij} = 1 & \qquad \forall j \in B \\ x_{ij} \in \{0, 1\} & \end{aligned}$$

In the LP relaxation (P), we allow $x_{ij} \geqslant 0$.

The set of feasible solutions to (P) form a polytope. The optimum for the linear objective over a polytope is an extreme point.

Theorem 2.1 (Egerváry)**.** *Any extreme point of (P) is a* 0-1 *vector.*

Corollary 2.2. *The minimum cost of a perfect matching is equal to the maximum weight of a c-vertex cover, where a c-vertex cover is a vector $y \in \mathbb{R}^v$ such that $y_i + y_j \leqslant c_{ij}$ $\forall i \in A, j \in B$.*

Proof of Theorem 2.1. Suppose there exist $u_i, v_j \in \mathbb{R}$ $\forall i \in A, j \in B$ such that $u_i + v_j \leqslant c_{ij}$. Then for a perfect matching M,

$$\sum_{i \in A, j \in B} c_{ij} \leqslant \sum_{i \in A} u_i + \sum_{j \in B} v_j. \tag{1}$$

Then we get a lower bound for (P) by (D):

$$\max \sum_{i\in A, j\in B} u_i + v_j$$
$$u_i + v_j \leqslant c_{ij} \qquad \forall i \in A, j \in B$$

Then $\min_{M \text{ perfect matching}} \sum_{ij\in M} c_{ij}x_{ij} \geqslant \min_{x\in(P)} \sum_{i\in A, j\in B} c_{ij}x_{ij} \geqslant \max_{(u,v)\in(D)} \sum_{i\in A} u_i + \sum_{j\in B} v_j$.

So given $(u, v) \in (D)$, a perfect matching M has equality in (1) if it contains only those edges ij with $c_{ij} = u_i + v_j$. □

Algorithm

Start with some feasible solution to (D)
repeat
Π-iteration: Find a maximum-cardinality matching M in $H = (A \cup B, E)$ where
$E = \{ij : i \in A, j \in B, c_{ij} = u_i + v_j\}$
if $|M| = n$ STOP and **return** M
Δ-iteration: Let R be the set of reachable vertices as before.
Recall that there is no edge in H between $A \cap R$ and $B - R$
$c_{ij} > u_i + v_j \ \forall i \in A \cap R, j \in B - R$
So let $\delta = \min_{i\in A\cap R, j\in B-R}\{c_{ij} - u_i - v_j\}$ and update:

$$u_i = \begin{cases} u_i + \delta & \text{if } i \in A \cap R \\ u_i & \text{if } i \in A - R \end{cases}, \; v_j = \begin{cases} v_j - \delta & \text{if } j \in B \cap R \\ v_j & \text{if } j \in B - R \end{cases}.$$

The updated (u, v) is feasible for (D) [Exercise]. Furthermore,

$$\begin{aligned}\left(\sum u_i + \sum v_j\right)_{\text{new}} - \left(\sum u_i + \sum v_j\right)_{\text{old}} &= \delta(|A \cap R| - |B \cap R|) \\ &= \delta(|A \cap R| + |A - R| - |A - R| - |B \cap R|) \\ &= \delta(|A| - |S|) = \delta(n - |M|) > 0\end{aligned}$$

So if the algorithm terminates, M is a minimum cost perfect matching.

To show that the algorithm terminates:
Claim 2.3. *After every Δ-iteration, $|B \cap R|$ increases.*

All previously reachable vertices are still reachable and a new edge ij, $i \in A\cap R$, $j \in B - R$ has $c_{ij} = u_i + v_j$ [Exercise], after n Δ-iterations, $|M|$ increases, so there are $O(n^2)$ iterations before termination. So the algorithm terminates in $O(n^4)$ time. It is actually possible to get $O(n^3)$.

Lecture 3, September 1, 2015

1 Polyhedral Theory

See Pages 1-3 of the "Basics of Polyhedral Theory" handout.

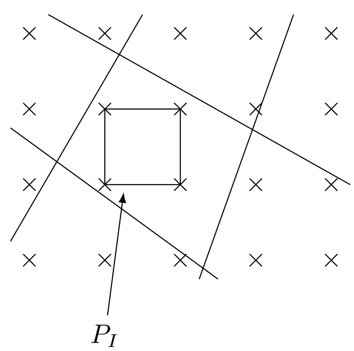
Lecture 4, September 3, 2015

1 Polyhedral Theory, continued

See Page 4 of the "Basics of Polyhedral Theory" handout.

2 Integer Programming

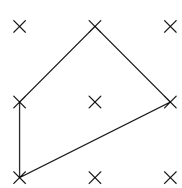
Definition 2.1. The *integral hull* of a polytope $P \subseteq \mathbb{R}^n$ is $P_I = \text{convhull}(P \cap \mathbb{Z}^n)$.
Example 2.2.



Theorem 2.3. *If P is a rational polyhedron then P_I is also a polyhedron or polytope.*

Note that $\max\{c^T x : x \in P \cap \mathbb{Z}^n\} = \max\{c^T x : x \in P_I\}$.

It is unclear how to get an inequality description of the integral hull.
Definition 2.4. A polyhedron P is an integer polyhedron if $P = P_I$.
Example 2.5.



These can be solved by LP techniques, so it is important to recognize when LPs are integer.
Theorem 2.6 (Edmonds-Giles)**.** *The following are equivalent:*

1. *P is integral, $P = P_I$.*
2. *$z = \max\{c^T x : x \in P\}$ is integral for all $c \in \mathbb{Z}^n$ and the value is bounded.*
3. *Every minimal face has an integral point.*
4. *$\max\{c^T x : x \in P\}$ has an integer optimal solution for all $c \in R^n$ when the value is bounded.*
5. *$\max\{c^T x : x \in P\}$ has an integral optimal solution for all $c \in \mathbb{Z}^n$ when the value is bounded.*

Note that none of these are efficient characterizations. A sufficient condition is that the matrix is unimodular:

Definition 2.7. A matrix A is *totally unimodular (TU)* if every square submatrix of A has determinant -1, 0, or 1.

Lecture 5, September 8, 2015

1 Total Unimodularity

Recall:
Definition 1.1. A matrix A is *totally unimodular (TU)* if every square submatrix of A has determinant -1, 0, or 1.
Lemma 1.2. *If A is TU and U is a nonsingular square submatrix of A, then U^{-1} is integral.*

Proof. By Cramer's rule:

$$|U^{-1}[i,j]| = \frac{|\det(U_{\overline{\text{ij}}})|}{|\det(U)|}$$

where $|\det(U_{\overline{\text{ij}}})| \in \{0,1\}$ and $|\det(U)| = 1$, so $|U^{-1}[i,j]| \in \{0,1\}$. □

Theorem 1.3. *If A is TU then for all $b \in \mathbb{Z}^m$, $P(b) = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is integral.*

Proof. Fix $b \in \mathbb{Z}^m$. Let F be a minimal face of $P(b)$, $F \neq \varnothing$. Then $F = \{x \in \mathbb{R}^n : A'x = b'\}$ where $A'x \leqslant b'$ is a subsystem of $Ax \leqslant b$. A' has full rank, so $A' = \begin{bmatrix} U & V \end{bmatrix}$ where U is square and has full rank. Since U is a submatrix of A, $\det(U) \in \{\pm 1\}$. So $x = \begin{bmatrix} U^{-1}b' \\ 0 \end{bmatrix} \in F \cap \mathbb{Z}^n$: $x \in F$ since $A'x = \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} U^{-1}b' \\ 0 \end{bmatrix} = b$, and $x \in \mathbb{Z}^n$ since U^{-1}, b are integral. So F contains an integral point. □

Some properties of TU matrices:

$$\begin{aligned} A \text{ is TU} &\iff A^T \text{ is TU} \\ &\iff \begin{bmatrix} A & I \end{bmatrix} \text{ is TU} \\ &\iff \begin{bmatrix} A & -A \end{bmatrix} \text{ is TU (in general, } \begin{bmatrix} A & -A & I & -I \end{bmatrix} \text{ is TU} \end{aligned}$$

Corollary 1.4. *A is TU $\implies \forall a,b,c,d$ integral, $P(a,b,c,d) = \{x \in \mathbb{R}^n, a \leqslant x \leqslant b, c \leqslant Ax \leqslant d\}$ is integral.*
Theorem 1.5. *The following are equivalent:*

1. *$A \in \mathbb{R}^{m \times n}$ is TU.*
2. *$\forall b \in \mathbb{Z}^m$, $P(b) = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is integral.*
3. *$\forall a,b,c,d \in \mathbb{Z}^m$, $P(a,b,c,d) = \{x \in \mathbb{R}^n, a \leqslant x \leqslant b, c \leqslant Ax \leqslant d\}$ is integral.*
4. *Every subset S of columns of A can be split into two sets S_1 and S_2 such that*

$$\sum_{i \in S_1} i - \sum_{j \in S_2} j \in \{0, \pm 1\}^m.$$

(1) $\iff$ (2) is due to [Hoffman-Kruskal], and (1) $\iff$ (4) is due to [Ghouila-Houri].

2 Applications of Total Unimodularity

2.1 Bipartite Matching

Theorem 2.1. *Let $A \in \{0, \pm 1\}^{m \times n}$ be such that each column of A contains at most two non-zero entries and there exists a partition $M_1 \cup M_2 = [m]$ of rows such that for each column j, $\sum_{i \in M_1} A_{ij} = \sum_{i \in M_2} A_{ij}$. Then A is TU.*

Proof. By induction. Let A be the smallest counterexample. Since A is the smallest, we can assume that A is square, and $\det A \notin \{0, \pm 1\}$. If a column of A is 0, then $\det A = 0$. If a column has only one nonzero entry, we can expand along that entry and obtain a smaller counterexample. Hence every column of A has exactly two nonzero entries, and $\sum_{i \in M_1} A_{ij} - \sum_{i \in M_2} A_{ij} = 0$ implies that the rows of A are linearly dependent, so $\det A = 0$. So no smallest counterexample exists. $\square$

2.1.1 Perfect Matching in Bipartite Graphs

Recall the LP relaxation:

$$\begin{aligned} \min \ & \sum c_{ij} x_{ij} \\ & \sum_{j \in B} x_{ij} = 1 \qquad \forall i \in A \\ & \sum_{i \in A} x_{ij} = 1 \qquad \forall j \in B \\ & x_{ij} \geqslant 0 \end{aligned}$$

In the corresponding constraint matrix M, rows correspond to vertices and columns correspond to edges, and $M[u, e] = \begin{cases} 1 & \text{if } e \sim u \\ 0 & \text{otherwise} \end{cases}$. So the number of nonzero entries in a column is exactly 2: one in A and one in B. This immediately suggests a partition $M_1 = A$ and $M_2 = B$. Hence M is TU. Since $b \in \{0, 1\}^m$, the LP-relaxation has an integral optimum.

Theorem 2.2. *The convex hull of incidence vectors of a perfect matching of a bipartite graph is*

$$\begin{aligned} & \operatorname{convhull}(\{x \in \{0,1\}^E : \operatorname{supp}(X) \textit{ is a perfect matching in } G\}) \\ & \qquad = \left\{ x \in \mathbb{R}^E : \sum_{i \in A} x_{ij} = 1 \ \forall j \in B, \sum_{j \in B} x_{ij} = 1 \ \forall i \in A, x \geqslant 0 \right\}. \end{aligned}$$

This implies the following:

Theorem 2.3 (Birkhoff-von Neumann)**.** *Let $A \in \mathbb{R}^{n \times n}$ be doubly stochastic (that is, it has non-negative entries and every row and column sum to 1). Then A can be expressed as a convex combination of permutation matrices.*

2.1.2 b-matchings ($b \in \mathbb{Z}_+^V$)

Definition 2.4. A *b-matching* is a set of edges such that the number of edges adjacent to v is at most $b(v)$.

A maximum weight b-matching satisfies the Integer Program

$$\begin{aligned}
\max \sum_{e\in E} w_e x_e & & \\
\sum_{e\in\delta(v)} x_e \leqslant b(v) & & \forall v \in V \\
x_e \in \{0,1\} & & \forall e \in E
\end{aligned}$$

The LP relaxation has integral optimum in bipartite graphs.

In summary:
Theorem 2.5. *A graph is bipartite iff its vertex-edge incidence matrix is TU.*

2.2 Network Flows

Theorem 2.6. *Let $A \in \{0, \pm 1\}^{m\times n}$, each column of A has at most one $+1$ and at most one -1. Then A is TU.*

Proof. By smallest counterexample. If a column has at most one nonzero then expand along that column. If all columns have two nonzeros then use Theorem 2.1 with $M_1 = [m]$, $M_2 = \varnothing$. ☐

In Maximum s-t flow, we are given a directed graph $D = (V, E)$, $c : E \to \mathbb{R}_+$, $s, t \in V$. To simplify the problem, we add an arc $t \rightsquigarrow s$ with $c_{ts} = \infty$.

We have the LP:

$$\begin{aligned}
\max \quad & x_{ts} \\
& \sum_{e\in\delta^+(v)} x_e - \sum_{e\in\delta^-(v)} x_e = 0 \qquad 0 \leqslant x_e \leqslant c_e
\end{aligned}$$

Lemma 2.7. *The constraint matrix M corresponding to a flow conservation is TU.*

Proof. In M, the rows correspond to vertices and columns correspond to arcs,

$$M[u,e] = \begin{cases} 1 & e \text{ is outgoing to } u \\ -1 & e \text{ is incoming to } u \\ 0 & \text{otherwise} \end{cases}$$

So M is TU. ☐

Theorem 2.8 (Ford-Fulkerson (1956))**.** *Given $D = (V, E)$, $c : E \to \mathbb{R}_+$, $s, t \in V$, the maximum value of a s-t flow is equal to the minimum capacity of an s-t cut.*

Also, if $c : E \to \mathbb{Z}_+$, then there exists an integral max flow [Dantzig].

If we impose lower bounds $\ell : E \to \mathbb{R}_+$ on the capacities, then the maximum s-t flow value is equal to

$$\min_{\substack{U\subseteq V\\ s\in U, t\notin U}} \left(\sum_{e\in\delta^+(U)} c(e) - \sum_{e\in\delta^-(U)} \ell(e) \right)$$

.

This implies the following:

Theorem 2.9 (Dilworth)**.** *Take a poset* $(P, \succeq)$*. Then the maximum size of an antichain is equal to the minimum number of chains needed to cover all elements of* P*.*

Actually, there exists an efficient characterization of TU matrices [Seymour]. In fact, there is a $O((m+n)^3)$ algorithm [Truemper].

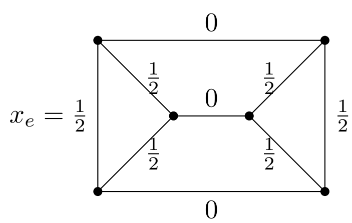
Lecture 6, September 10, 2015*

1 Nonbipartite Matchings

Definition 1.1. The perfect matching polytope of $G = (V, E)$ is $PM(G)$ = the convex hull of indicator vectors of perfect matchings in G.

We saw that if G is bipartite, then $PM(G)$ is given by $Q(G) = \{x \in \mathbb{R}^E : x(\delta(v)) = 1\ \forall v \in V, x \geqslant 0\}$, where $x(T) = \sum_{e \in T} x_e$.

It is easy to see that these constraints are not enough in nonbipartite graphs.
Example 1.2.



This is an extreme point of $Q(G)$, but in any matching, it is clear that we should pick at least one of the horizontal edges.

In general, in an odd set at last one vertex should be matched outside, so:

$$P(G) = \{x \in \mathbb{R}^E : x(\delta(v)) = 1\ \forall v \in V, x \geqslant 0, x(\delta(U)) \geqslant 1\ \forall U \subseteq V, |U| \geqslant 3, |U| \text{ odd}\}.$$

Theorem 1.3 (Edmonds (1965))**.** $PM(G) = P(G)$.

Proof. $PM(G) \subseteq P(G)$ is easy.

$P(G) \subseteq PM(G)$: We need to show that vertices in $P(G)$ are integral. Suppose not. Pick a smallest counterexample, which is a graph $G = (V, E)$ with minimum $|V| + |E|$. There exists an extreme point $x \in P(G)$, x fractional. Since x is extreme it cannot be written as a convex combination.

Claim 1.4. *$|V|$ is even.*

Proof. $P(G)$ is nonempty. ■

Claim 1.5. $0 < x_e < 1\ \forall e \in E$

Proof. If $x_e = 0$, we can delete e. If $x_e = 1$, we can remove both endpoints and get a smaller counterexample. ■

Claim 1.6. $\deg(v) \geqslant 2\ \forall v \in V$.

Proof. If $\deg v = 1$, $x_e = 1$ for the unique edge attached to it. ■

*Updated Feb 1, 2018

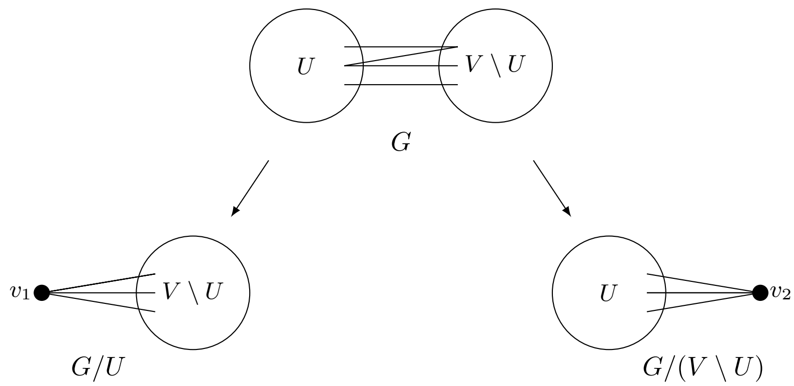
So $|E| \geqslant |V|$.

Claim 1.7. $|E| > |V|$

Proof. $2|E| \geqslant \sum_{v \in V} \deg v \geqslant 2|V|$, so if $|E| = |V|$ then G is a disjoint union of cycles, but every cycle must be even, and so G is bipartite and $P(G) \subseteq PM(G)$. ■

So x is a vertex of $P(G)$, and so there are $|E|$ tight constraints determining x, and $|E| > |V| \implies \exists U \subseteq V, |U| \text{ odd}, |U| \geqslant 3, x(\delta(U)) \geqslant 1$.

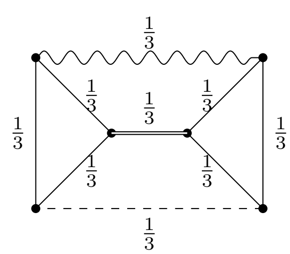
Now contract, keeping all edges as parallel if necessary:

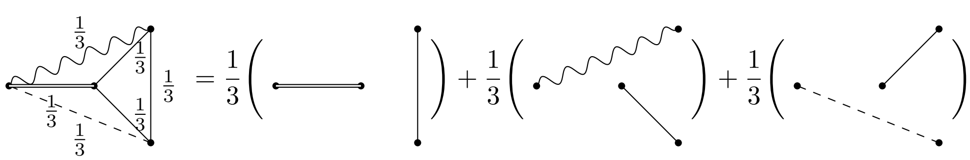


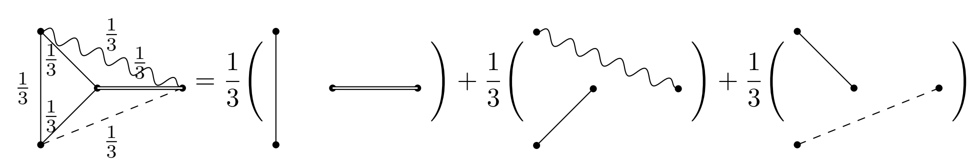
Denote $x^i = x|_{E(G_i)}$.

Let us proceed by example, to see where we are going with this.

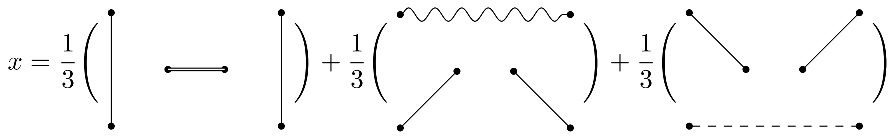
Example 1.8.







We can then glue these together as follows:



contradicting that x is extreme.

We resume the proof.

Claim 1.9. $x^i \in P(G_i)$ *for* $i = 1, 2$.

Proof. We prove it for x^1; the proof for x^2 is similar.
If $v \in V(G_1)$, $v \neq v_1$ implies $x^1(\delta(v)) = x(\delta(v)) = 1$, and $v = v_1$ implies $x^1(\delta(v)) = x(\delta(v)) = 1$. Let $S \subseteq V(G_1)$, $|S|$ odd.
If $S \subseteq V \setminus U$, we are done since all edges leaving S survive the contraction, $x^1(\delta(S) = x(\delta(S)) \geqslant 1$.
Otherwise, $v_1 \in S$, then $x^1(\delta(S) = x(\delta(S \cup U) \geqslant 1$ since $S \cup U$ is odd. ■

But $|V(G^i)| + |E(G^i)| < |V| + |E|$, so by minimality of the counterexample,

$$\begin{cases} x^1 = \sum_{i=1}^{\ell_1} a_i \chi^{M_i(G_1)}, \sum_{i=1}^{\ell_1} a_i = 1, a_i \geqslant 0 \\ x^2 = \sum_{i=1}^{\ell_2} b_i \chi^{M_i(G_2)}, \sum_{i=1}^{\ell_1} b_i = 1, b_i \geqslant 0 \end{cases}$$

for perfect matchings $M_i(G_1)$, $M_i(G_2)$ in G_1 and G_2, respectively.

Since x is rational, x_1 and x_2 are rational. So there exist p_i, q_i, k such that $a_i = p_i/k$, $b_i = q_i/k$.

By repeating matchings, we may as well take

$$\begin{cases} x^1 = \sum_{i=1}^{k} \frac{1}{k} \chi^{M_i(G_1)} \\ x^2 = \sum_{i=1}^{k} \frac{1}{k} \chi^{M_i(G_2)} \end{cases}$$

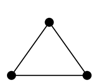
Consider $e \in \delta(U)$ such that $x_e = \frac{p}{k}$. Then exactly p perfect matchings $M_i(G_1)$ use e and exactly p perfect matchings $M_i(G_2)$ use e. We then pair them up and glue to obtian p perfect matchings M_i' in G. (In the example the matchings were already conveniently in this form.)

Doing this for all edges gives $x = \sum_{i=1}^{k} \frac{1}{k} \chi^{M_i'}$, contradiction. □

2 Matching Polytope

Definition 2.1. The matching polytope $P_{\text{mat}}(G)$ is the convex hull of indicator vectors of matchings in G.

This is not the same as $PM(G)$.
Example 2.2.



In this example, $PM(G)$ is empty, but $P_{\text{mat}}(G)$ is not:
Theorem 2.3. $P_{mat}(G) = \left\{ x \in \mathbb{R}^E : x(\delta(v)) \leqslant 1\ \forall v \in V, x(E[U]) \leqslant \left\lfloor \frac{|U|}{2} \right\rfloor \forall U \notin V, |U|\ odd, x \geqslant 0 \right\}$.

Lecture 7, September 15, 2015

Recall $P_{\text{mat}}(G)$ is the convex hull of the indicator vectors of matchings in G, and

$$R(G) = \left\{ x \in \mathbb{R}^E : x(\delta(v)) \leqslant 1 \ \forall v \in V, x(E[U]) \leqslant \left\lfloor \frac{|U|}{2} \right\rfloor \forall U \notin V, |U| \text{ odd}, x \geqslant 0 \right\}.$$

Theorem 0.4 (Edmonds)**.** *$P_{mat}(G) = R(G)$.*

Proof. $P_{\text{mat}}(G) \subseteq R(G)$ is easy. To show $R(G) \subseteq P_{\text{mat}}(G)$:

Let $x \in R(G)$.

Let G' be a copy of G. Define $\tilde{G}$ by $V(\tilde{G}) := V(G) \cup V(G')$, $E(\tilde{G}) = E(G) \cup E(G') \cup \{uu' : u \in V(G)\}$, with

- $\tilde{x}_{uv} = x_{uv} \ \forall uv \in E(G)$
- $\tilde{x}_{u'v'} = x_{uv} \ \forall u'v' \in E(G)$
- $\tilde{x}_{uu'} = 1 - x(\delta(u)) \ \forall u \in V(G)$

Claim 0.5. *$\tilde{X} \in PM(\tilde{G})$.*

This would imply that $\tilde{x}$ is a convex combination of perfect matchings in $\tilde{G}$, and hence by projection, x is a convex combinations of matchings in G.

Proof of Claim. First, note that $\tilde{x}(\delta_{\tilde{G}}(v)) = 1 \ \forall v \in V, \tilde{x} \geqslant 0$ (Exercise).
Let $U \subseteq V(\tilde{G})$, $|U|$ odd.

Claim 0.6. *If $U \subseteq V(G)$, $\tilde{x}(\delta_{\tilde{G}}(v)) \geqslant 1$.*

Proof. We have $\sum_{v \in V} \tilde{x}(\delta_{\tilde{G}}(v)) = |U|$ which implies

$$\tilde{x}(\delta_{\tilde{G}}(U)) + 2\tilde{x}(E_{\tilde{G}}[U]) = |U|$$

which implies

$$\tilde{x}(\delta_{\tilde{G}}(U)) = |U| - 2\tilde{x}(E_{\tilde{G}}[U]) \geqslant |U| - 2\left(\frac{|U|-1}{2}\right) = 1. \quad \blacksquare$$

We have a similar result if $U \subseteq V(G')$.
If $U = S \cup T'$, $S \subseteq V(G), T' \subseteq V(G')$, then

$$|U| = |S \setminus T| + |T \setminus S| + 2|S \cap T|.$$

If $|U|$ is odd, at least one of the first two terms is odd. WLOG let $|S \setminus T|$ be odd.

Claim 0.7. *$\tilde{x}(\delta_{\tilde{G}}(S \cup T')) \geqslant \tilde{x}(\delta_{\tilde{G}}(S \setminus T))$ ($\geqslant 1$ by the previous claim)*

Proof. The only edges in $\delta_{\tilde{G}}(S \setminus T)$ but not in $\delta_{\tilde{G}}(S \cup T')$ are edges from $S \setminus T$ to $S \cap T$. But these have a "mirror" from $S' \cap T'$ to $S' \setminus T'$, which contributes to $\tilde{x}(\delta_{\tilde{G}}(S \cup T'))$. $\blacksquare$

$\blacksquare$

$\square$

$R(G)$ can be encoded by the LP $P(w)$:

$$\begin{array}{ll} \max & \sum_{e\in E} w_e x_e \\ & x(\delta(v) \leqslant 1 \qquad \forall v \in V \\ & x(E[V]) \leqslant \left\lfloor \frac{|U|}{2} \right\rfloor \quad \forall U \subseteq V, |U| \text{ odd}, |U| \geqslant 3 \end{array}$$

hence it suffices to solve this LP, but this LP is exponentially sized. So a natural question is, is this the smallest LP to describe the matching polytope? This was resolved last year: it is impossible to reduce to less than subexponential.

Let's derive a minimax relation. This comes from duality. The dual $D(w)$:

$$\begin{array}{ll} \min & \sum_{u\in V} Y_u + \sum_{\substack{U\subseteq V \\ |U| \text{ odd} \\ |U|\geqslant 3}} Z_U \lfloor \frac{|U|}{2} \rfloor \\ & Y_u + Y_v + \sum_{\substack{U\subseteq V \\ |U| \text{ odd} \\ |U|\geqslant 3}} Z_U \geqslant w_e \qquad \forall e = uv \in E \\ & Y \geqslant 0 \\ & Z \geqslant 0 \end{array}$$

It turns out that the dual has integral solutions, which will lead to an "odd cover" of the graph.

Definition 0.8. An *odd cover* of $G = (V, E)$ is a subset $S \subseteq V$ and a collection $U_1, \ldots, U_k$ of odd sets such that every edge e has at least one endpoint in S or both its endpoints in U_i for some $i \in [k]$. The weight of an odd cover $(S, U_1, \ldots, U_k)$ is $|S| + \sum_{i=1}^k \lfloor \frac{|U_i|}{2} \rfloor$.

Definition 0.9 (Edmonds-Giles)**.** A system $Ax \leqslant b$ is *totally dual integral* (TDI) if $\forall c \in \mathbb{Z}^n$ such that $\max\{c^T x : Ax \leqslant b\}$ is finite, the dual $\min\{y^T b : y^T A = c^T, y \geqslant 0\}$ has an integral optimal solution.

Theorem 0.10. *Let $P = \{x : Ax \leqslant b\}$ be a polytope, with $Ax \leqslant b$ TDI and b integral. Then P is an integral polytope.*

Proof. Recall a polytope P is integral iff for all integral w, the optimal value of $\max\{w^T x : x \in P\}$, if finite, is integer. Let us write the primal and dual LPs:

$$\begin{array}{ll} \max & w^T x \\ & Ax \leqslant b \end{array} \qquad \begin{array}{ll} \min & y^T b \\ & y^T A = w^T \\ & y \geqslant 0 \end{array}$$

Note that integral w achieves integer minimum value (by TDI), which implies primal achieves integer maximum value for all integral w, which implies $P = \{x : Ax \leqslant b\}$ is an integral polytope. $\square$

We show that the matching polytope is TDI.

Theorem 0.11 (Cunningham-Marsh)**.** *The system*

$$\begin{array}{ll} x(\delta(v) \leqslant 1 & \forall v \in V \\ x(E[V]) \leqslant \left\lfloor \frac{|U|}{2} \right\rfloor & \forall U \subseteq V, |U| \text{ odd}, |U| \geqslant 3 \\ x \geqslant 0 & \end{array}$$

is TDI.

They actually showed a stronger result.

Definition 0.12. A family F of sets is *laminar* if $\forall U, W \in F$, $U \cap W = \varnothing$ or $U \subseteq W$ or $W \subseteq U$.

Theorem 0.13. *If w is integral then there exists (y, z) optimal to $D(w)$ such that (y, z) is integral and $\{U \subseteq V, |U| \text{ odd}, Z_U \geqslant 0\}$ is laminar.*

In the proof we use that $P(w)$ has an integral optimal solution.

Proof. By induction on the $|E| + \sum_{e\in E} w_e$:

Pick a counterexample $(G = (V, E), w)$ with minimum $|E| + \sum_{e\in E} w_e$. This implies that $D(w)$ has no integral solution, and w is integral.

Claim 0.14. $w_e \geqslant 1$ $\forall e \in E$.

Proof. Otherwise we can discard w_e and get a smaller counterexample. ■

Claim 0.15. *$\forall v \in V$, there exists a maximum w-weight matching that exposes v.*

Proof. By contradiction, suppose not. Then $\exists v \in V$ covered by all maximum w-weight matchings.
Set w' to be obtained from w by subtracting one from every edge incident to v.
If the weight does not decrease, then v is not chosen by any maximum w-weight matching.
If the weight of the maximum w'-weight matching is smaller than the weight of the maximum w-weight matching by 1, then this gives a smaller instance. By minimality of the counterexample, $\exists(\bar{y}, \bar{z})$ integral that is optimal for $D(w')$.
Set $y_u = \begin{cases} \bar{y}_u & \text{if } u \neq v \\ \bar{y}_u + 1 & \text{if } u = v \end{cases}$, and $z_u = z$. Then (y, z) is feasible for $D(w)$ (exercise).
Then $\sum_{u\in V} y_u + \sum_{\substack{U\subseteq V \\ |U| \text{ odd} \\ |U|\geqslant 3}} z_U$ is equal to the weight of the maximum w-weight matching. Thus (y, z) is an integral optimal solution for $D(w)$, a contradiction. ■

Claim 0.16. *Dual optimal solutions for $D(w)$ are of the form $(Y = 0, Z)$.*

Proof. Let Y, Z be optimal to $D(w)$. Fix $v \in V$. Then there exists a maximum w-weight matching exposing v. By complementary slackness, $Y_v = 0$ $\forall v$. ■

Among all dual optimal solutions $(0, Z)$, pick one that maximizes $\sum_{\substack{U\subseteq V \\ |U| \text{ odd} \\ |U|\geqslant 3}} Z_U \left\lfloor \frac{|U|}{2} \right\rfloor^2$.

Claim 0.17. *Let $F := \{U \subseteq V : |U|$ odd$, Z_U > 0\}$. Then F is laminar.*

[We ended here. We will continue in the next lecture.] □

Lecture 8, September 17, 2015

1 Finishing the proof from last time

Recall the LP $P(w)$:

$$\begin{array}{lll} \max & \sum_{e \in E} w_e x_e & \\ & x(\delta(v) \leqslant 1 & \forall v \in V \\ & x(E[V]) \leqslant \left\lfloor \frac{|U|}{2} \right\rfloor & \forall U \subseteq V, |U| \text{ odd}, |U| \geqslant 3 \end{array}$$

and the dual $D(w)$:

$$\begin{array}{ll} \min & \sum_{u \in V} Y_u + \sum_{\substack{U \subseteq V \\ |U| \text{ odd} \\ |U| \geqslant 3}} Z_U \left\lfloor \frac{|U|}{2} \right\rfloor \\ & Y_u + Y_v + \sum_{\substack{U \subseteq V \\ |U| \text{ odd} \\ |U| \geqslant 3}} Z_U \geqslant w_e \quad \forall e = uv \in E \\ & Y \geqslant 0 \\ & Z \geqslant 0 \end{array}$$

We were proving the theorem

Theorem 1.1. *If w is integral then there exists (y, z) optimal to $D(w)$ such that (y, z) is integral and $\{U \subseteq V, |U|$ odd$, Z_U \geqslant 0\}$ is laminar.*

Proof. We picked the smallest counterexample $(G = (V, E), w)$ with minimum $|E| + \sum_{e \in E} w_e$. w is integral and $D(w)$ does not have integral solutions. We argued that:

1. $w_e \geqslant 1\ \forall e \in E$
2. $\forall v \in V$, there is a maximum w-weight matching exposing v
3. As a consequence, all optimal solutions of $D(w)$ are of the form $(Y = 0, Z)$.

Among all optimal solutions to $D(w)$ pick Z that maximizes the potential function $\sum_{\substack{U \subseteq V \\ |U| \text{ odd} \\ |U| \geqslant 3}} Z_U \left\lfloor \frac{|U|}{2} \right\rfloor^2$.

Claim 1.2. *Let $F := \{U \subseteq V : |U|$ odd$, Z_U > 0\}$. Then F is laminar.*

Proof. Suppose it is not laminar. Then there must be two sets T, W that intersect and are not completely contained in each other: $\exists v \in T \cap W$, $T \setminus W, W \setminus T \neq \varnothing$.
Let M be a maximum w-weight matching that exposes v. The goal is to argue that we can move weights away from T and W, to $T \cap W$ and $T \cup W$, to get another solution that improves on the potential. In order to move weights away we need to show that $T \cap W$ and $T \cup W$ are odd sets.
Note that $T, W \in F$, so $Z_T, Z_W > 0$. The dual variables are greater than 0, so the primal constraints should be tight by complementary slackness. This implies that $x^M(E[T]) = \lfloor \frac{|T|}{2} \rfloor$ and $x^M(E[W]) = \lfloor \frac{|W|}{2} \rfloor$. This means that all of the vertices of T, except for one, is matched within. This is also true for W. So all vertices within the intersection should be matched within the intersection, and all vertices outside the intersection should be matched outside the intersection. Then v is the only vertex exposed by M in $G[T]$ and $G[W]$, and so $|T \cap W|$, $|T \cup W|$ are odd.
Let $\varepsilon = \min\{Z_T, Z_W\}$, then for $U \subseteq V$, $|U|$ odd, $|U| \geqslant 3$, set

$$Z'_U = \begin{cases} Z_U - \varepsilon & \text{if } U = T \text{ or } W \\ Z_U + \varepsilon & \text{if } U = T \cap W \text{ or } T \cup W \\ Z_U & \text{otherwise} \end{cases}$$

Then $(0, Z')$ is a feasible solution for $D(w)$, and the objective value of $(0, Z')$ is equal to the objective value of $(0, Z)$ [Exercise]. So $(0, Z')$ is optimal for $D(w)$. Let us compute the change in potential:

$$\sum_{\substack{U \subseteq V \\ |U| \text{ odd} \\ |U| \geqslant 3}} \left\lfloor \frac{|U|}{2} \right\rfloor^2 (Z'_U - Z_U) = \varepsilon \left(\left\lfloor \frac{|T \cap W|}{2} \right\rfloor^2 + \left\lfloor \frac{|T \cup W|}{2} \right\rfloor^2 - \left\lfloor \frac{|T|}{2} \right\rfloor^2 - \left\lfloor \frac{|W|}{2} \right\rfloor^2 \right) > \varepsilon$$

a contradiction. ■

Finally we show:

Claim 1.3. *Z is integral.*

Proof. We will use the fact that Z is laminar. Suppose Z is not integral. Pick the largest $T \in F$ such that Z_T is fractional. Let $U_1, \ldots, U_k$ be the maximal sets in F contained in T, and $\varepsilon = Z_T - \lfloor Z_T \rfloor$. Now we obtain another solution by pushing ε into the inner sets:

$$\overline{Z}_U = \begin{cases} Z_U - \varepsilon & \text{if } U = T \\ Z_U + \varepsilon & \text{if } U = U_i \\ Z_U & \text{otherwise} \end{cases}$$

We show that $\overline{Z}$ is feasible to $D(w)$. To do this, we argue that the constraints are satisfied.
So look at an edge inside some U_i. Since Z_T is decreased but Z_{U_i} is increased by the same amount, if the edge was satisfied before, it is still satisfied.
Now look at an edge not contained in any U_i or has one endpoint in U_i and one endpoint outside. This edge was oversatisfied since w is integral and Z_T is fractional, so we just took away the fractional portion. So the edge is still satisfied.
Finally,

$$\text{obj-value}(0, \overline{Z}) - \text{obj-value}(0, Z) = \varepsilon \left(\sum_{i=1}^{k} \left\lfloor \frac{|U_i|}{2} \right\rfloor - \left\lfloor \frac{|T|}{2} \right\rfloor \right) < -\varepsilon$$

contradicting the optimality of Z. ■

□

Corollary 1.4. *The maximum weight of a w-weight matching is exactly equal to the minimum weight of an odd w-cover.*

2 Separation for the Perfect Matching Polytope

Recall that optimization over P is to maximize a linear objective w. By the ellipsoid algorithm, this is equivalent to finding a separation over P: given v, find z such that $z^T x < z^T \ \forall x \in P$, or show that $v \in P$. We do not know how to do this for matching directly, but we can do this using a separation for $PM(G)$.

Recall the perfect matching polytope

$$PM(G) = \{x \in \mathbb{R}^E : x(\delta(v)) = 1 \ \forall v \in V, x \geqslant 0, x(\delta(U)) \geqslant 1 \ \forall U \subseteq V, |U| \text{ odd}\}$$

Given $G = (V, E)$, $X : E \to \mathbb{R}_+$, $|V|$ even, the goal is to find $U \subseteq V$, $|U|$ odd, $x(\delta(U)) < 1$ or prove that no such U exists.

The approach is to treat X as weights on the edges and find a minimum weight cut T. If $|T|$ turns out to be odd, we are done. If $|T|$ is even, then...

This motivates the minimum-weight odd T-cut problem.
Definition 2.1. For $T \subseteq V$, an odd T-cut is a subset $U \subseteq V$ such that both $U \cap T$ and $\overline{U} \cap T$ are odd sets. For $X : E \to \mathbb{R}$, then the weight of an odd T-cut U is $x(\delta(U))$.

So let us focus on the minimum-weight odd T-cut problem [Padberg-Rao]. We are given $G = (V, E)$, $X : E \to \mathbb{R}_+$, and $T \subseteq V$, $|T|$ even. The goal is to find a minimum weight odd T-cut.

Here is how we solve this. First, find a minimum-weight cut U of V. If $|U \cap T|$ is odd, then we are done. Otherwise, we will argue that there is a minimum-weight odd cut that is entirely contained in U or $\overline{U}$, and then we can recurse. Before we do that, let us show a simpler claim.
Claim 2.2. *Suppose we have $C, D \subseteq V$, $C, D \neq \varnothing$, $X : E \to \mathbb{R}_+$. Then*

$$x(\delta(C \cap \overline{D})) + x(\delta(\overline{C} \cap D)) \leqslant x(\delta(C)) + x(\delta(D)).$$

Proof. Exercise. □

Lemma 2.3. *Suppose we have $G = (V, E)$, $X : E \to \mathbb{R}_+$, $T \subseteq V$, $|T|$ even. Let $S \subseteq V$ be such that:*

(i) $S \cap T \neq \varnothing$, $\overline{S} \cap T \neq \varnothing$

(ii) $x(\delta(S))$ is minimal among all such sets satisfying (i).

Then there exists a minimum weight odd T-cut U such that $U \subseteq S$ or $U \subseteq \overline{S}$.

Proof. If $S \cap T$ is odd, we are already done. Assume $S \cap T$, $\overline{S} \cap T$ are even.

Say W is a minimum-weight odd T-cut. Without loss of generality, assume that $|T \cap W|$ is odd. Then since $|T \cap W| = |T \cap W \cap S| + |T \cap W \cap \overline{S}|$, one of the terms is odd, so say $|T \cap W \cap S|$ is odd, then $|T \cap W \cap \overline{S}|$ is even. But $|S \cap T|$ is even, so $|S \cap T \cap \overline{W}|$ is odd. Next, T is even, so $|\overline{S} \cap T \cap \overline{W}|$ is even.

Note that since $\overline{S} \cap T \neq \varnothing$, so $\overline{S} \cap W \cap T$ and $\overline{S} \cap \overline{W} \cap T$ cannot both be empty. Without loss of generality, assume $\overline{S} \cap W \cap T \neq \varnothing$. Observe that $\overline{(\overline{S} \cap W)} \cap T = (S \cup |W|) \cap T \supseteq S \cap T \neq \varnothing$, so $\overline{S} \cap W$ satisfies (i) and so by (ii),

$$x(\delta(S) \leqslant x(\delta(\overline{S} \cap W)). \tag{1}$$

Now $S \cap \overline{W} \cap T$ is odd, so $\overline{(S \cap \overline{W})} \cap T = (\overline{S} \cup W) \cap T$ is odd, so $S \cap \overline{W}$ is an odd T-cut. Since W is a minimum-weight odd T-cut,

$$x(\delta(W)) \leqslant x(\delta(S \cap \overline{W})). \tag{2}$$

Adding (1) and (2) gives

$$x(\delta(S) \leqslant x(\delta(\overline{S} \cap W)) + x(\delta(W)) \leqslant x(\delta(\overline{S} \cap W)) + x(\delta(S \cap \overline{W})).$$

But by Claim 2.2, the reverse inequality holds, so (1) and (2) are tight, so $S \cap \overline{W}$ is a minimum weight odd T-cut, so setting $U = S \cap \overline{W}$ ends the proof. □

We will use this lemma to obtain an algorithm.

Algorithm

1 Base Case: If $|T| = 2$, say $T = \{s, t\}$, then return the minimum s-t cut.
2 Find $S \subseteq V$ such that $S \cap T \neq \varnothing$, $\overline{S} \cap T \neq \varnothing$, $X(\delta(S))$ is minimal among all such sets S
3 **if** $|S \cap T|$ is odd **return** S
4 Let G_1 be a graph obtained by contracting $\overline{S}$ to a single vertex and $T_1 := T \cap S$.
  Similarly, let G_2 be obtained by contracting S to a single vertex and $T_2 := T \cap \overline{S}$.
5 Recurse on (G_1, T_1) and (G_2, T_2) and **return** the smaller of the two cuts.

The correctness of the algorithm follows directly from the lemma. The running time is as follows: Fix $u \in T$, and assume that $u \in S \cap T$. Then there should be some vertex $v \in \overline{S} \cap T$, so S should also be a minimum weight u-v cut for some v. So Line 2 corresponds to $|T|$ minimum s-t cuts. In each recursion, the size of T will reduce by at least 1. So the number of recursive calls is at most T. So the algorithm has $O(|T|^2)$ minimum s-t cuts.

So to separate over $PM(G)$, we would just set $T = V$ and solve the minimum weight odd T-cut problem.

We can use this to design a separation for the matching polytope using the reduction we defined before.
Question 2.4. Design a separation algorithm for $P_{\mathrm{mat}}(G)$ directly.
Question 2.5. Given $G = (V, E)$, $X : E \to \mathbb{R}_+$, $T_1, T_2 \subseteq V$, $|T_1|, |T_2|$ even, find S that is an odd T_1-cut and odd T_2-cut minimizing $x(\delta(S))$.

Lecture 9, September 22, 2015

1 Maximum Cardinality Matching

We are only interested in non-bipartite graphs, as we have already seen everything for bipartite graphs.
Definition 1.1. For $G = (V, E)$, let $\nu(G)$ be the cardinality of a maximum matching in G. For $U \subseteq V$, let $\theta_G(U)$ be the number of connected components in $G - U$ with an odd number of vertices.

Let M be a matching. Fix $U \subseteq V$. Let us look at the matching restricted to each of the connected components in $G - U$. In each odd component $S \subseteq V$ in $G - U$, the restriction $M[S]$ exposes at least one vertex. Such vertices have to be matched by M to vertices in U. Thus the number of M-exposed vertices is at least $\theta_G(U) - |U|$. Note that $|M| = \frac{1}{2}(|V| - \#$ of M-exposed vertices). This tells us that $|M| \leqslant \frac{1}{2}(|V| - |U| - \theta_G(U))$. Hence for any graph G, $\nu(G) \leqslant \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - \theta_G(U))$. It turns out that this last inequality is tight.
Theorem 1.2 (Tutte-Berge)**.** $\nu(G) = \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - \theta_G(U))$.

Note that this is a min-max relation.

Proof. We will show equality by induction on $|V|$. So let us pick a counterexample $G = (V, E)$ with smallest $|V|$. This means that $\nu(G) < \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - \theta_G(U))$.

Claim 1.3. *G is connected.*

Proof. Say $G = G_1 \dot\cup G_2$. The minimality of the counterexample implies that there exists $U_i \subseteq V(G_i)$, with $\nu(G_i) = \frac{1}{2}(|V(G_i)| + |U_i| - \theta_{G_i}(U_i))$. Then setting $U = U_1 \cup U_2$, then $|U| = |U_1| + |U_2|$ and $\theta_G(U) = \theta_{G_1}(U_1) + \theta_{G_2}(U_2)$. But

$$\begin{aligned}
\nu(G) &= \nu(G_1) + \nu(G_2) \\
&= \frac{1}{2}(|V(G_1)| + |V(G_2)| + |U_1| + |U_2| - \theta_{G_1}(U_1) - \theta_{G_2}(U_2)) \\
&= \frac{1}{2}(|V| + |U| - \theta_G(U))
\end{aligned}$$

contradicting that G is a counterexample. ■

Claim 1.4. *$\forall v \in V$, there exists a maximum cardinality matching exposing v.*

Proof. Say v is matched by every maximum cardinality matching. Let $G' = G - v$. By minimality of the counterexample, there exists $U' \subseteq V(G')$ such that $\nu(G') = \frac{1}{2}(|V(G')| + |U_i| - \theta_{G'}(U'))$.
So take $U = U' \cup \{v\}$. Then $|U| = |U'| + 1$, and $\theta_G(U) = \theta_{G'}(U')$. Since $\nu(G) = \nu(G') + 1$, we obtain

$$\begin{aligned}
\nu(G) &= \frac{1}{2}(|V(G')| + |U'| - \theta_{G'}(U')) + 1 \\
&= \frac{1}{2}(|V| - 1 + |U'| - 1 - \theta_G(U)) + 1 \\
&= \frac{1}{2}(|V| + |U| - \theta_G(U))
\end{aligned}$$

contradicting that G is a counterexample. ■

This means that $\nu(G) \leqslant \frac{|V|-1}{2}$.

Claim 1.5. *$\nu(G) = \frac{|V|-1}{2}$ and $|V|$ is odd.*

Proof (Lovasz). Suppose the claim does not hold. Then every maximum cardinality matching exposes at least two vertices. Pick a maximum matching M that exposes u, v such that the distance between u and v is minimal. Note that $\mathrm{dist}_G(u, v) > 1$.
Let P be a shortest path between u and v. From $\mathrm{dist}_G(u, v) > 1$ we know that there must be an intermediate vertex in P. Since u and v are the closed M-exposed vertex, every $t \in V(P) - \{u, v\}$ should be matched by M. Fix $t \in V(P) - \{u, v\}$.
Let M' be a maximal matching exposing t. M' should necessarily match both u and v (otherwise (M', t, v) or (M', t, u) would violate the choice of (M, u, v)). Among all such M' pick the one that maximizes $|M \cap M'|$.
Note that $|M'| = |M|$, so M' exposes another vertex $s \notin \{u, v, t\}$ that is matched by M. In fact, s is not in the path P (otherwise (M', s, t) violates the choice of (M, u, v)).
Let y be the vertex matched to s by M. Then $\exists yz \in M'$ (otherwise $M' \cup \{sy\}$ is a larger matching). Then $N = (M' - yz) \cup \{sy\}$ is a matching, $|N| = |M'|$, N exposes t, matches u and v, and $|N \cap M| \geqslant |M' \cap M|$, so N contradicts the choice of M'. ■

This completes the proof of the theorem as taking $U = \varnothing$ gives $\frac{|V|+|U|-\theta_G(U)}{2} = \frac{|V|-1}{2} = \nu(G)$, contradicting that G is a counterexample. □

A set $U \subseteq V$ attaining the minimum in the Tutte-Berge formula is called a Tutte set or Tutte witness.
Corollary 1.6. *Let M be a maximum maching, $U \subseteq V$ be a set attaining the minimum in the Tutte-Berge formula. Then*

1. *M has exactly $\lfloor \frac{|V(S)|}{2} \rfloor$ edges in every component S of $G - U$*
2. *all vertices of U are matched to vertices in some odd component of $G - U$*

Proof. Suppose S is an odd component in $G - U$ and say $|M[S]| < \frac{|V(S)|-1}{2}$. So at least 3 vertices of S are not matched within S. So the number of M-exposed vertices is at least $\theta_G(U) - 1) + 3 - |U|$, so $|M| \leqslant \frac{1}{2}(|V| + |U| - \theta_G(U) - 2)$, a contradiction.

The proof is similar for even components and the second part. □

Theorem 1.7 (Tutte's 1-factor)**.** *G has a perfect matching iff the number of odd components in $G - U$ is at most $|U|$ for all $U \subseteq V$.*

Proof.

$$\begin{aligned}
G \text{ has a perfect matching} &\iff \nu(G) = \frac{1}{2}|V| \\
&\iff \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - \theta(U)) = \frac{1}{2}|V| \\
&\iff \frac{1}{2}(|V| + |U| - \theta_G(U)) \geqslant \frac{|V|}{2} \quad \forall U \subseteq V \\
&\iff \theta_G(U) \leqslant |U| \quad \forall U \subseteq V
\end{aligned}$$

□

This is a generalization of Hall's theorem.

2 Algorithm for maximum cardinality matching [Edmonds]

Recall the following:
Lemma 2.1. *M is a maximum matching in G iff there does not exist any M-augmenting path in M.*

We used this lemma to find a maximum matching in bipartite graphs.

Since the lemma is true for any graph, we will look for M-augmenting paths again. The goal is to find such a path efficiently. In bipartite graphs, we were able to do this using breadth-first search or depth-first search starting at an exposed vertex. But in non-bipartite graphs, we can start and end at the same vertex. Such paths do not give M-augmenting paths. Note that such paths must be caused by odd loops.

Lecture 10, September 24, 2015

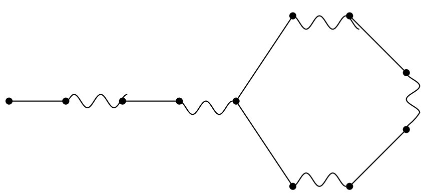
1 An algorithm for non-bipartite maximum cardinality matching [Edmonds]

Recall the following lemma:
Lemma 1.1. *M is a maximum matching in G iff there does not exist any M-augmenting path in M.*

We used this lemma to find a maximum matching in bipartite graphs.

Since the lemma is true for any graph, we will look for M-augmenting paths again. The goal is to find such a path efficiently. In bipartite graphs, we were able to do this using breadth-first search or depth-first search starting at an exposed vertex. But in non-bipartite graphs, we can start and end at the same vertex. Such paths do not give M-augmenting paths.

The following is a basic example of such a case:



Edmonds had the idea that if we see one of these "balloons", we will just shrink them. We will see why this idea works.
Definition 1.2. A *walk* in G is a finite sequence of vertices $v_0, v_1, v_2 \ldots, v_t$ such that $v_i v_{i+1} \in E \ \forall i = 0, 1, \ldots, t-1$. The *length* of such a walk is t.

A walk can repeat vertices whereas a path cannot.
Definition 1.3. Let M be a matching in G. A walk $(v_0, v_1, \ldots, v_t)$ is an *M-alternating walk* if exactly one of $v_{i-1}v_i$ and $v_i v_{i+1}$ belongs to M $\forall i = 1, \ldots, t-1$.

We will now formally describe the scenario pictured above:
Definition 1.4. An *M-flower* is an M-alternating walk $v_0, v_1, \ldots, v_t$ such that:

1. $v_0 \in X$ (where X is the set of exposed vertices)
2. $v_0, v_1, \ldots, v_{t-1}$ are distinct
3. t is odd
4. $v_t = v_i$ for some even $i < t$

The cycle $(v_i, v_{i+1}, \ldots, v_t)$ is called an *M-blossom.* The vertex v_i is the *base* of the M-blossom.
Fact 1.5. *In an M-blossom based at v, every vertex $u \neq v$ has an even-length M-alternating path to v.*

Let us prove the correctness of the algorithm.
Definition 1.6. Let $B \subseteq V$ be a blossom. Then G/B is the graph obtained by contracting B to a new vertex called B;

- $V(G/B) = V(G \setminus B) \cup \{B\}$
- $E(G/B) = E(G \setminus B) \cup \{uB : \exists uv \in E \text{ with } u \in V \setminus B, v \in B\}$

Let M/B denote the image of M on G/B.

[**TODO** Figure goes here]

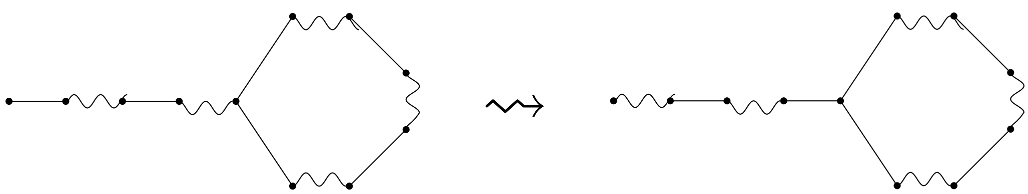
Note that we retain parallel edges.

Theorem 1.7. *Let B be a M-blossom in G. M is a maximum matching in G iff M/B is a maximum matching in G/B.*

This is the reason why we can shrink such structures and continue searching in the smaller graph.

Proof. ($\Longrightarrow$): By contradiction. Suppose M/B is not a maximum matching in G/B. Then there exists an M/B-augmenting path P in G/B. We may assume that P goes through the contracted vertex B, since otherwise P would be an M-augmenting path in G. Recall that only the base of the blossom is matched outside, and we know that every other vertex in the blossom has an even-length M-alternating path to the base. So P can be extended to an M-augmenting path in G. Contradiction.

($\Longleftarrow$): By contradiction again. Suppose that M is not a maximum matching in G. Then there exists an M-augmenting path in G. We may assume that the base of B is exposed by M, since otherwise we can just do the following simple transformation:



M $\quad\quad\quad\quad$ M'

Then M is a maximum matching in G iff M' is a maximum matching in G.

So we need to show that M'/B is a maximum matching in $G/B \implies M'$ is a maximum matching in G.

We are assuming that M' is not a maximum matching in G, so let $P = (u_0, u_1, \ldots, u_s)$ be an M'-augmenting path in G. If P does not intersect B, then P is also an M'/B-augmenting path in G/B. So P intersects B. Note that the blossom has exactly one M'-exposed vertex, but P has two, so at least one of u_0, u_s is not in B. Without loss of generality, let $u_0 \notin B$. Let u_j is the first vertex in P that intersects B. Then $(u_{0,1}, \ldots, u_{j-1}, B)$ is an M'/B-augmenting path. Contradiction. ☐

Theorem 1.8. *Let $P = v_0, v_1, \ldots, v_t$ be a shortest M-alternating x-x walk (a walk between two exposed vertices). Then either P is an M-augmenting path or $v_0, v_1, \ldots, v_j$ is an M-flower for some $j \leqslant t$.*

Proof. Assume P is not an M-augmenting path. We would like to find a pre-sequence that forms an M-flower. Choose the smallest j such that $v_j = v_i$ for some $i < j$. So $v_0, v_1, \ldots, v_{j-1}$ are distinct.

If v_i to v_j is an even cycle, then we can shortcut to obtain a shorter M-alternating x-x walk.

Now suppose v_i to v_j is an odd cycle. Suppose i is odd. Then $v_i = v_j$ is matched to both v_{i+1} and v_{j-1}. So $v_{i+1} = v_{j-1}$ This contradicts that v_j is the first repeated vertex.

So i is even. Then $v_0, \ldots, v_j$ is an M-flower. ☐

Finally, the algorithm:

Algorithm:

Given: Matching M, Graph G
Goal: Find M-augmenting path, if any

if $\exists$ an x-x M-alternating walk
    Find a shortest such walk P
    if P is an M-augmenting path
        return P
    else
        Let B be a blossom in the M-flower present in P
        Apply the algorithm recursively to M/B in G/B to obtain an M/B-augmenting path Q
        Expand Q to an M-augmenting path in G
        return the path
else
    Stop

The correctness follows from the two theorems.

The running time of the algorithm is as follows: most of the work is in finding P. The way we do this is to create a directed graph D containing two copies of G with cross edges such that if $uv \in M$, then we put directed edges $u'v$ and $v'u$, otherwise we put uv' and vu'. So we can just find the shortest path using whatever-first-search on D. If it finds a blossom, it must recurse. The depth of the recursion is $|V|$. So the total running time is $O(|V|^2|E|)$.

The current best known algorithm takes time $O(|E|\sqrt{|V|})$ [Micali-Vazirani].

Lecture 11, September 29, 2015

Recall

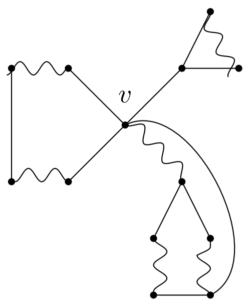
$$\nu(G) = \min_{U \subseteq V} \frac{1}{2}(|V| + |U| - \theta_G(U))$$

where $\theta(U)$ is the number of odd components in $G - U$.

Definition 0.9. A set U that achieves the minimum on the right-hand side is known as the *Tutte set* or *Tutte witness.*

Recall that if U is a Tutte set, then a maximal matching matches any vertex of U to an odd component.

Example 0.10.

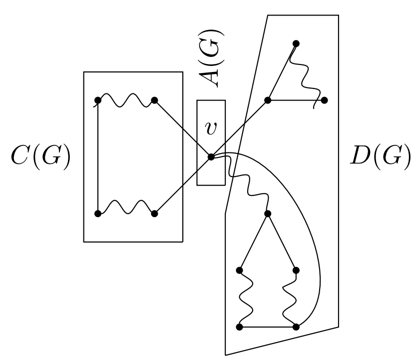


Note that $\nu(G) = 6$. If $U = \{v\}$ then $\theta(U) = 2$, so $\frac{1}{2}(|V| + |U| - \theta_G(U)) = \frac{13+1-2}{2} = 6$, so U is a Tutte set. On the other hand if $U = \varnothing$ then $\frac{1}{2}(|V| + |U| - \theta_G(U)) = \frac{13+0-1}{2} = 6$ as well. So the Tutte set is not unique.

The following is due to [Edmonds-Gallai] Let $G = (V, E)$.

- $D(G) := \{v \in V : \exists \text{ a maximum matching exposing } v\}$
- $A(G) := N(D(G)) = \{u \in V : \exists uv \in E, v \in D(G)\}$
- $C(G) := V \setminus (A(G) \cup D(G))$

Example 0.11.



Note that we said that $A(G) = \{v\}$ is a Tutte set. In general, this is true:

Theorem 0.12. *Let $U = A(G)$.*

(i) U is a Tutte set.

(ii) $D(G)$ is the union of the odd components in $G - U$.

(iii) $C(G)$ is the union of the even components of $G - U$.

Lemma 0.13. *Let M be a maximum matching in G, X be the M-exposed vertices.*

(i) $X \subseteq D(G)$.

(ii) $v \in D(G) \iff \exists$ an M-alternating path between v and X.

Proof. (i) is by definition.

(ii): ($\Longleftarrow$): Let P be an M-alternating path between v and x. If P exposes v we are done, otherwise $N = M \triangle E(P)$ is a maximum matching exposing v, so $v \in D(G)$.

($\implies$): Let $v \in D(G)$. Then there exists a maximum matching N exposing v. Look at $M \triangle N$. This is a union of even-length M-alternating paths and cycles. So there should be a path P that starts at v alternating between edges in M and edges in N between v and some $u \in X$. □

Proof of Theorem 0.12. We consider two cases.

Case 1: D is an independent set. We would like to show that $A(G)$ is a Tutte set. Let M be a maximum matching in G, X be the M-exposed vertices. We know that M matches every vertex in $A(G)$ (by definition of $A(G)$). Let $v \in A(G)$, so there exists some $uv \in M$.

Claim 0.14. $u \in D(G)$.

Proof. By contradiction. Suppose $u \notin D(G)$. But $v \in A(G)$ means that there is some $w \in D(G)$ which is adjacent to v, so let P be an M-alternating path between w and X, say, $P = v_0, v_1, \ldots, v_t = w$ with $v_0 \in X$. We know that $v_0 \in D(G)$ and $w \in D(G)$. But for each even vertex v_i in P, we can flip the edges before v_i to get a maximum matching exposing v_i. So $v_i \in D(G)$ for all even i.
If $uv \in E(P)$, then either v is an even vertex and $v \in D(G)$ or u is even and $u \in D(G)$, a contradiction. So $uv \notin E(P)$. Now let us extend P by w, v, u to obtain Q. Q is an M-alternating path from X to u, so $u \in D(G)$. Contradiction. ■

Note that this claim holds even when $D(G)$ is not an independent set.

Now let us prove the three statements.

We have
$$\theta(U) \geqslant |D(G)| = |X| + |A(G)| = |V| - 2|M| + |U| \tag{1}$$
so $|M| \geqslant \frac{1}{2}(|V| + |U| - \theta(U))$. Recall that $|M| \leqslant \frac{1}{2}(|V| + |U| - \theta(U))$ holds for all U. So we have an equality and U is a Tutte set. This means (1) is tight, so $\theta(U) = D(G)$; this gives (ii) and (iii).

Case 2: $D(G)$ is not an independent set. We will use induction on $|V|$. The idea is that we will pin down an M-blossom B. Then we will apply the induction hypothesis to G/B, then we will lift back up.

Say that there exists $uv \in E$, $u, v \in D(G)$. Then let M, N be maximum matching exposing u, v respectively. Neither matching should match both, otherwise they would not be maximal. Consider $M \triangle N$, which is a union of even length M-alternating paths and cycles. So there exists an M-alternating path P starting at u in $M \Delta N$. If P does not end in v, then $P \cup \{uv\}$ is an N-augmenting path. So P ends in v. But then $P \cup \{uv\}$ is an M-blossom B based at u.

Let $G' = G/B$, $M' = M/B$, X' be the set of M'-exposed vertices in G'. We know that $|M'| = \nu(G')$ (from last lecture).

Claim 0.15. $D(G') = (D(G) \setminus B) \cup \{\bar{B}\}$ *where $\bar{B}$ is the contracted vertex of B.*

Proof. First, $\bar{B} \in D(G')$ since M' exposes $\bar{B}$. Next, let $v \in V \setminus B$. By Lemma 0.13,

$$\begin{aligned} v \in D(G) &\iff \exists\ M\text{-alternating path between } v \text{ and } X \text{ in } G \\ &\iff \exists\ M\text{-alternating path between } v \text{ and } X' \text{ in } G' \text{ [Exercise]} \\ &\iff v \in D(G'). \end{aligned} \qquad \blacksquare$$

So $B \subseteq D(G)$, and $A(G) = A(G')$, $C(G) = C(G')$.

By induction, $D(G')$ is the union of odd components in $G' - A(G')$. This also means that $D(G)$ is the union of odd components in $G - A(G)$. This proves (ii) and in turn proves (iii). Furthermore, we know $|M'| = \frac{1}{2}(|V(G')| + |A(G')| - \theta_{G'}(A(G')))$. Then

$$|M| = \frac{|V| - |V'|}{2} + |M'| = \frac{1}{2}(|V| + |A(G')| + \theta_{G'}(A(G'))) = \frac{1}{2}(|V| + |A(G)| + \theta_G(A(G)))$$

so $A(G)$ is a Tutte set. $\square$

Next lecture, we will see why this decomposition can be found efficiently, and what information it gives us about the maximum matchings in G.

Lecture 12, October 1, 2015

1 More on the Edmonds-Gallai decomposition

We continue with the Edmonds-Gallai decomposition:

Let $G = (V, E)$. We define:

- $D(G) := \{v \in V : \exists \text{ a maximum matching exposing } v\}$
- $A(G) := N(D(G)) = \{u \in V : \exists uv \in E, v \in D(G)\}$
- $C(G) := V \setminus (A(G) \cup D(G))$

We showed:
Theorem 1.1 (Edmonds-Gallai)**.**

(i) $A(G)$ *is a Tutte set, i.e.* $\frac{|V|+|A(G)|-\theta(A(G))}{2} = \nu(G)$

(ii) $D(G)$ *is the union of the odd components in* $G - A(G)$*.*

(iii) $C(G)$ *is the union of the even components of* $G - A(G)$*.*

Definition 1.2. (D, A, C) is known as the *Edmonds-Gallai decomposition* of G.

The Edmonds-Gallai decomposition tells us a lot about the structure of a maximum matching.
Corollary 1.3. *Let* M *be a maximum matching in* G*. Then*

(i) $M[C(G)]$ *is a perfect matching in* $G[C(G)]$

(ii) $M[K]$ *exposes one vertex in* K *for each component* K *in* $G[D(G)]$

(iii) M *matches all vertices in* A *to distinct components of* $D(G)$

Definition 1.4. A graph $G = (V, E)$ is *factor-critical* (f.c.) if $\forall v \in V$, the graph $G - v$ has a perfect matching.
Example 1.5. C_{2n+1} and K_{2n+1} are factor-critical.

Note that G is f.c. $\implies \nu(G) = \frac{|V(G)|-1}{2} \implies |V(G)|$ is odd. Also, G is f.c. $\implies$ G is connected.
Corollary 1.6. *Let* $G = (V, E)$*. Then each component of* $G[D(G)]$ *is f.c.*

Proof. Let K be a component of $G[D(G)]$. Let $u \in V(K)$. We need to show that removing u leaves a perfect matching in K. Well, $u \in D(G)$, so there exists a maximum matching M that exposes u. Recall the corollary of the Tutte-Berge formula that M has $\frac{|V(K)|-1}{2}$ edges in K, so $M[K]$ is a perfect matching in $K - u$. $\square$

Corollary 1.7. *A connected graph* G *is f.c. iff* $U = \varnothing$ *is the unique Tutte set in* G*.*

So the question is how do we obtain the decomposition. It is sufficient to compute $D(G)$, but how do we do this?

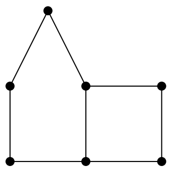
Note that $v \in D(G) \iff \nu(G) = \nu(G - v)$. So we can go through each vertex, compute $\nu(G - v)$ using Edmonds' algorithm, and then compare to $\nu(G)$. So the Edmonds-Gallai decomposition can be computed in $O(|V|^{\frac{3}{2}}|E|)$.

Given a maximum matching, we can in fact obtain (D, A, C) in $O(|V|^2)$ time by tracing M-alternating paths.

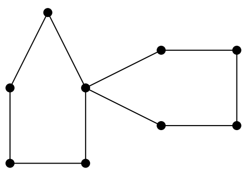
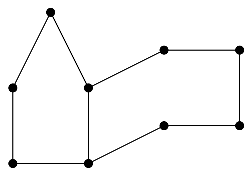
Now given (D, A, C), how do we obtain a maximum matching? First, we would delete A and D and find a perfect matching in C. Next, shrink each component in D and delete edges within A; this gives a bipartite graph, so find a maximum matching. Next, re-expand the components of D; in each component matched with A, removed the vertex matched with a vertex in A, and in the other components, remove an arbitrary vertex (since the component is f.c.). Then find a perfect matching within the remaining vertices of D.

2 More on factor-critical graphs

We said that odd cycles are f.c. Now what about the following graph?



The answer is yes. So are the following:

 and 

In all of these examples we started with an odd cycle and added an odd-length path (called an *ear*) to it. In fact every f.c. graph is obtained in this way. For example, K_{2n+1} is obtained from C_{2n+1} by adding edges, which are odd-length paths.

Definition 2.1.

1. A graph H arises from G *by adding an ear* if H is obtained from G by adding a path between two (not necessarily distinct) vertices of G, i.e. there exists a path $P = v_0, v_1, \ldots, v_t$ in H with $v_1, \ldots, v_{t-1}$ having degree 2 and $G = H - \{v_1, \ldots, v_{t-1}\}$.
2. P is known as *an ear*
3. An ear is *odd* if it has odd length
4. An ear is *proper* if v_0 and v_t are distinct
5. A sequence of graphs $G_0, G_1, \ldots, G_k = G$ is an *ear decomposition of* G if $\forall i = 1, \ldots, k$, G_i arises from G_{i-1} by adding an ear.

Theorem 2.2 (Whitney)**.** *G is 2-vertex-connected iff there exists a proper ear decomposition of G from a cycle.*

Theorem 2.3 (Robbins)**.** *G is 2-edge-connected iff there exists an ear decomposition of G from a cycle.*

Theorem 2.4 (Lovasz)**.** *G is f.c. iff G has an odd ear decomposition starting from a single vertex.*

Proof. ($\Longleftarrow$): By induction on the number of ears. The base case is trivial. Assume that G_i is f.c., and G_{i+1} arises from G_i by adding $P = v_0, \ldots, v_t$.

Case 1: $u \in V(G_i)$, then there is a perfect matching M_i in $G_i - u$, so take $M_{i+1} = M_i \cup \{v_1v_2, \ldots, v_{t-2}v_{t-1}\}$. This is a perfect matching in $G_{i+1} - v$.

Case 2: $u \in \{v_1, \ldots, v_{t-1}\}$. Say $u = v_j$. If j is even, then let M_i be a perfect matching in $G_i - v_0$, then take $M_{i+1} = M_i \cup \{v_0v_1, \ldots, v_{j-2}v_{j-1}, v_{j+1}v_{j+2}, \ldots, v_{t-2}v_{t-1}\}$. This is a perfect matching in $G_{i+1} - v_j$. If j is odd, take M_i to be a perfect matching in $G_i - v_t$, and take $M_{i+1} = M_i \cup \{v_1v_2, \ldots, v_{j-2}v_{j-1}, v_{j+1}v_{j+2}, \ldots, v_{t-1}v_t\}$.

($\implies$): We will construct an ear decomposition. First, G is connected. Let $u \in V$. Let M_u be a perfect matching in $G - u$. Pick the largest subgraph H such that (1) H has odd ear decomposition from u, and (2) no edge of M_u crosses H, i.e. $M_u \cap \delta(V(H), V(G \setminus H)) = \varnothing$. Such an H exists, e.g. $H = (\{u\}, \varnothing)$. If $E(G) = E(H)$, we are done.

Suppose $E(H) \neq E(G)$. Now G is connected, so there should be at least one edge vw, $v \in H$, $w \in G \setminus H$. Let M_w be a perfect matching in $G - w$. The only even path in $M_u \triangle M_w$ starts at w and ends at u.

[We ended here. We will continue next lecture.] $\square$

Lecture 13, October 6, 2015

1 Finishing up from last time

We were in the middle of proving the following theorem:
Theorem 1.1 (Lovasz)**.** *G is f.c. iff G has an odd ear decomposition starting from a single vertex.*

Proof (continued). ($\implies$): We will construct an ear decomposition. Let $u \in V$, and M_u be a perfect matching in $G - u$. Pick the largest subgraph H such that (1) H has an odd ear decomposition from u, and (2) no edge of M_u crosses H, i.e. $M_u \cap \delta(V(H), V(G \setminus H)) = \varnothing$. Such an H exists, e.g. $H = (\{u\}, \varnothing)$. If $E(G) = E(H)$, we are done.

Suppose $E(H) \subsetneq E(G)$. Since G is connected, there exists $vw \in E$, $v \in V(H)$, $w \in V(G \setminus H)$. Let M_w be a perfect matching in $G - w$. Since M_u exposes only u and M_w exposes only w, the only even path in $M_u \triangle M_w$ starts at u and ends at w, so say $P = v_1, v_2, \ldots, v_t$ with $v_1 = w$ and $v_t = u$. Let $j > 1$ be the smallest index such that $v_j \in V(H)$. If j is even, then $v_{j-1}v_j \in M_u$, so M_u crosses H, a contradiction. So j is odd. But then the path $Q = v, v_1, \ldots, v_j$ is odd, hence $H' = H \cup Q$ arises from H by adding an odd ear and M_u does not cross H', so H' contradicts the maximality of H. ☐

Corollary 1.2. *Let $G = (V, E)$ with $|V| \geqslant 2$. G is f.c. iff there exists an odd cycle C such that G/C is f.c.*

2 Solutions to two HW problems

1. (a) If G is d-regular, bipartite, then G has a bipartite matching.

   Proof. Let $G = (A \sqcup B, E)$, $P(G) = \{x \in \mathbb{R}^E : x(\delta(u)) = 1\ \forall u \in A \sqcup B, x_e \geqslant 0\ \forall e \in E\}$. If $P(G)$ is nonempty, then it has an extreme point. We showed that $P(G)$ is integral, and an integral extreme point corresponds to the incidence vector of a perfect matching, in particular, there exists a perfect matching in G. So it suffices to show that $P(G) \neq \varnothing$. Note that $x_e = \frac{1}{d}\ \forall e \in E$ is in $P(G)$. This satisfies the constraints. ☐

2. Let G be bipartite. Let $Q(G)$ be the convex hull of matchings of size $\leqslant k$ and $P(G) = \{x \in \mathbb{R}^E : x(\delta(u)) \leqslant 1\ \forall u \in V, x_e \geqslant 0\ \forall e \in E, \sum_{e \in E} x_e \leqslant k\}$. Show that $Q = P(G)$.

   Proof Sketch. $Q(G) \subseteq P(G)$ is trivial.

   To show $P(G) \subseteq Q(G)$ it suffices to show that the extreme points of $P(G)$ are integral. The constraint matrix of $P(G)$ is TU, and the RHS b is integral. So $P(G)$ is integral. ☐

3 An algorithm for Maximum Weight Matching [Edmonds]

In fact, we will only focus on the Minimum Weight Perfect Matching problem, since we can reduce from Maximum Weight Matching to Minimum Weight Perfect Matching problem (see $\tilde{G}$ from Lecture 7).

We already saw one algorithm for Minimum Weight Perfect Matching, which was via separation.

The algorithm we will see today will come from Edmonds; this result predates the idea that separation implies matching. This is a primal-dual algorithm, like the Hungarian algorithm from Lecture 2.

Recall that in Minimum Weight Perfect Matching, we are given a graph $G = (V, E)$ and weights $w : E \to \mathbb{R}_{\geqslant 0}$. We would like to solve the LP and dual simultaneously.

Primal:

$$\begin{aligned}
\min \quad & \sum_{e \in E} w_e x_e \\
& x(\delta(v)) = 1 \quad \forall v \in V \\
& x(\delta(U)) \geqslant 1 \quad \forall U \subseteq V, |U| \text{ odd}, |U| \geqslant 3 \\
& x_e \geqslant 0 \quad \forall e \in E
\end{aligned}$$

Dual:

$$\begin{aligned}
\max \quad & \sum_{\substack{U \subseteq V \\ |U| \text{ odd}}} \pi(U) \\
& \sum_{\substack{U \subseteq V \\ |U| \text{ odd} \\ e \in \delta(U)}} \pi(U) \leqslant w_e \quad \forall e \in E \\
& \pi(U) \geqslant 0 \quad \forall U \subseteq V, |U| \text{ odd}, |U| \geqslant 3
\end{aligned}$$

The *Primal-Dual technique* works as follows:

- Maintain a dual feasible solution and an integer but *infeasible* primal solution.
- *Primal step*: try to find a primal solution that satisfies complementary slackness with the current dual. This will succeed if the current dual solution is optimal, and then we are done.
- If the primal step fails, then *Dual step*: modify the dual to improve its objective value.

The issue with this is that there are exponentially many odd subsets of V. What Edmonds did was a very cute trick. Recall the Cunningham-Marsh result, which showed that the primal is TDI and the optimal dual solution is supported on a Laminar family of sets. So we will only search over and maintain Laminar families of sets.

So we will maintain a dual feasible solution π supported on a laminar collection Ω of odd sets. Note that Ω includes singletons $\{v\}$. We set $\pi(U) = 0 \ \forall U \notin \Omega, U \subseteq V, |U|$ odd.

The following is an exercise:
Lemma 3.1. $|\Omega| \leqslant 2|V|$.

So the number of dual variables we need to maintain is at most $2|V|$, which is polynomial in the input size.

The algorithm is very similar to the cardinality matching algorithm.

Given $G = (V, E)$, Ω laminar, $\pi : \Omega \to \mathbb{R}$ dual feasible.
Definition 3.2.

(1) An edge e is *tight* if $\sum_{\substack{U \subseteq V \\ |U| \text{ odd} \\ e \in \delta(U)}} \pi(U) = w_e$.

(2) $E_\pi :=$ set of tight edges

(3) $G_\pi := (V, E_\pi)$

Suppose we find a perfect matching in G_π. Since the dual constraint is tight, the matching in G_π satisfies complementary slackness. So we will repeatedly update π, restrict to G_π, and look for a perfect matching of G_π.

Lecture 14, October 8, 2015

1 Continuing with the Algorithm for Minimum Cost Perfect Matching

Recall that given $G = (V, E)$, $w : E \to R_{\geqslant 0}$, and we had the LP and dual:

Primal:

$$\begin{aligned} \min \quad & \sum_{e \in E} w_e x_e \\ & x(\delta(v)) = 1 \quad \forall v \in V \\ & x(\delta(U)) \geqslant 1 \quad \forall U \subseteq V, |U| \text{ odd}, |U| \geqslant 3 \\ & x_e \geqslant 0 \quad \forall e \in E \end{aligned}$$

Dual:

$$\begin{aligned} \max \quad & \sum_{\substack{U \subseteq V \\ |U| \text{ odd}}} \pi(U) \\ & \sum_{\substack{U \subseteq V \\ |U| \text{ odd} \\ e \in \delta(U)}} \pi(U) \leqslant w_e \quad \forall e \in E \\ & \pi(U) \geqslant 0 \quad \forall U \subseteq V, |U| \text{ odd}, |U| \geqslant 3 \end{aligned}$$

We said that we would solve this using the complementary slackness conditions for (x, π):

- $x_e > 0 \implies e$ is tight for π
- $\pi(U) > 0 \implies x(\delta(U)) = 1$

Any optimal solution must satisfy both conditions.

Recall that we said that we will only maintain a laminar family of odd sets in the dual, since we know that the optimal solution is a laminar family.

Definition 1.1. Let Ω be laminar, $\pi : \Omega \to \mathbb{R}$ dual feasible, and:

(1) E_π be the tight edges

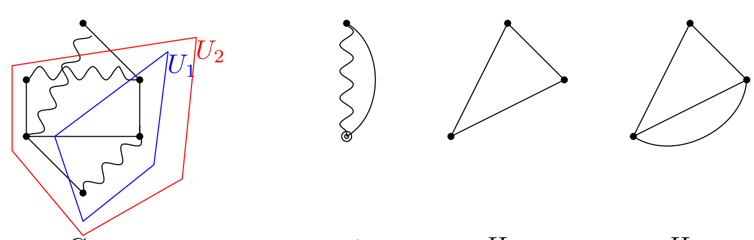
(2) $G_\pi = (V, E_\pi)$

(3) G' be the graph obtained from G by contracting all maximal sets in Ω to a vertex

(4) For each $U \in \Omega$, let H_U be the graph obtained from $G_\pi[U]$ by contracting all maximal proper subsets of U in Ω.

(5) For $v \in V(G')$, let S_v be the nodes in G contracted to v

Example 1.2.



G_π $\quad$ G' $\quad$ H_{U_1} $\quad$ H_{U_2}

$\Omega = \{U_1, U_2\} \cup \{\{v\} : v \in V\}$

The algorithm maintains

1. A laminar collection Ω of odd subsets of V
2. A dual feasible solution $\pi : \Omega \to \mathbb{R}$
3. A matching M in G'
4. A Hamiltonian cycle C_U in H_U $\forall U \in \Omega, |U| \geqslant 3$

1.1 The Algorithm

Input: Graph $G = (V, E)$ with edge-weights $w : E \to \mathbb{R}_{\geqslant 0}$
Output: A minimum w-weight matching M

(a) **Initialize**: $\Omega = \{\{v\} : v \in V\}$, $\pi(\{v\}) = 0$ for every $v \in V$, $M = \varnothing$

(b) **While (M is not a perfect matching in G'):**

Step 1. Find $X :=$ set of M-exposed vertices in G'

Step 2. Find a shortest M-alternating X-X walk P in G' (could be empty).

(i) If P is an M-augmenting path, then augment M and continue to Step 1.

(ii) If P has an M-blossom B then set

$$U := \bigcup_{v \in B} S_v,\ \Omega \leftarrow \Omega \cup \{U\},\ \pi(U) \leftarrow 0,\ G' \leftarrow G'/B,\ M \leftarrow M/B$$

Step 3. If no M-alternating X-X walk in G', then M is a maximum matching in G'.

(i) Find $R_{\text{odd}} := \{v \in V : \exists$ an odd length M-alternating v-X walk in $G'\}$
$R_{\text{even}} := \{v \in V : \exists$ an even length M-alternating v-X walk in $G'\}$

(ii) Let ε be the largest value such that the following modification to the dual solution π preserves feasibility:

$$\pi(U) \leftarrow \begin{cases} \pi(U) + \varepsilon & \text{if } U \in R_{\text{even}} \\ \pi(U) - \varepsilon & \text{if } U \in R_{\text{odd}} \end{cases}$$

If ε is unbounded, then G has no perfect matching. STOP.

(iii) If $\pi(U)$ decreases to zero for some $U \in R_{\text{odd}}$ with U being a contracted vertex, then remove U from Ω, update G_π and G', and extend M by a perfect matching in $C_U - v$ where v is the vertex of C_U covered by M.

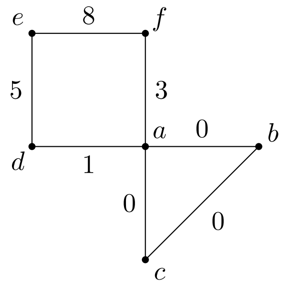
(c) **Repeat until Ω contains only singletons:**
For each maximal $U \in \Omega$ that is not a singleton:

- Remove U from Ω
- Exend M by a perfect matching in $C_U - v$, where v is the vertex of C_U covered by M.

(d) Output M

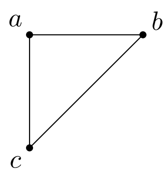
1.2 Example of the Algorithm

Example 1.3.



Initialize: $\Omega = \{\{v\} : v \in V\}$, $\pi(v) = 0$, $M = \varnothing$.

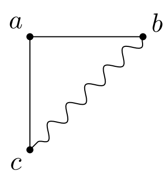
G_π:



(1) $X = V$

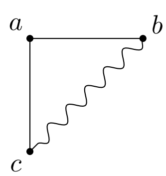
(2) (i) $P = b, c$

G_π:

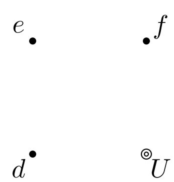


(ii) $U = \{a, b, c\}$, $\Omega = \{U\} \cup \{\{v\} : v \in V\}$, $\pi(U) = 0$

G_π:



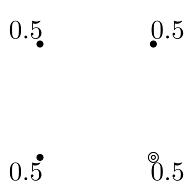
G':



(3) (i) $R_{\text{even}} = \{d, e, f, U\}$, $R_{\text{odd}} = \varnothing$

(ii) $\varepsilon = 0.5$

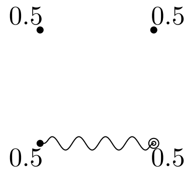
G':



(1) $X = \{d, e, f, U\}$

(2) $P = Ud$

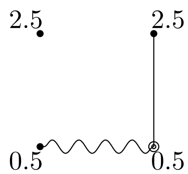
(i) G':



(3) (i) $R_{\text{even}} = \{e, f\}$, $R_{\text{odd}} = \varnothing$

(ii) $\varepsilon = 2$

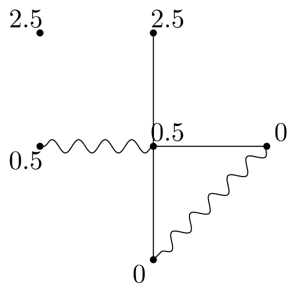
G':



(i) $R_{\text{even}} = \{d, e, f\}$, $R_{\text{odd}} = \{U\}$

(ii) $\varepsilon = 0.5$

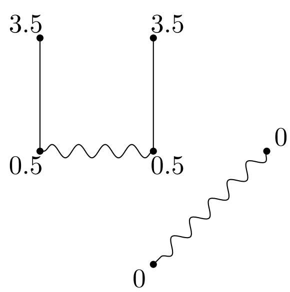
(iii) G':



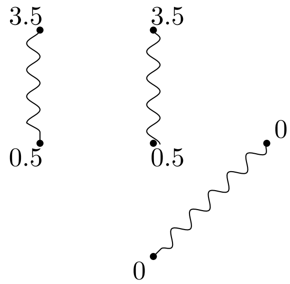
(i) $R_{\text{even}} = \{d, e, f\}$, $R_{\text{odd}} = \{a\}$

(ii) $\varepsilon = 0.5$

G':



(2) (i) G':



STOP.

1.3 Proof of Correctness

Lemma 1.4. *The algorithm indeed maintains the four invariants.*

Proof. Exercise. $\square$

Lemma 1.5. *If the algorithm terminates, then M is an optimum matching.*

Proof. Step (b) terminates $\implies$ we have a perfect matching N in G' $\implies$ we have a perfect matching M in G. Let π be a dual feasible solution at the end of the algorithm, and $x_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise} \end{cases}$ a primal feasible solution.

Then $\pi(U) > 0 \implies U \in \Omega \implies |N \cap \delta(U)| = 1 \implies |M \cap \delta(U)| = 1 \implies x(\delta(U)) = 1$.

On the other hand, $x_e > 0 \implies e \in M \subseteq E_\pi \implies e$ is tight $\implies (x, \pi)$ satisfies complementary slackness. $\square$

Note that we still have not proved that the algorithm terminates.

Lecture 15, October 13, 2015

1 Continuing with the Algorithm for Minimum Cost Perfect Matching

Lemma 1.1. *Step (b) terminates in $O(|V|^2)$ iterations.*

Proof. In each iteration we have a primal augmentation, dual shrinking, or deshrinking, or a new edge becomes tight. The number of primal augmentations is $\frac{|V|}{2}$. We will show that between every two primal augmentations, the algorithm can only perform $O(|V|)$ iterations.

Claim 1.2. *A set U that is shrunk will not be deshrunk before the next primal augmentation.*

Proof. Suppose U is shrunk. Then U is an M-blossom. So after contraction, $U \in R_{\text{even}}$, so until the next primal augmentation, either U remains in R_{even} or U is swallowed by a larger set U' that is shrunk. In either case, $\pi(U)$ does not decrease, which means U is not deshrunk. ■

Between two primal augmentations, the maximum number of deshrinkings is at most $|\Omega| \leqslant 2|V|$. The maximum number of shrinkings is at most $|V|$.

We have bounded the number of augmentations, shrinkings, and deshrinkings. So we still need to bound the number of times an edge can become tight. In fact the number of times this can happen is at most $2|V|$ [Exercise]. The idea is that either the sizes of $R_{\text{even}}, R_{\text{odd}}$ increase or it is followed by a deshrinking. □

Theorem 1.3. *The minimum cost perfect matching problem can be solved in $O(|V|^2|E|)$ time. The maximum weight matching problem can also be solved.*

Question 1.4. Can maximum weight matching be solved directly, i.e. without reducing to minimum cost perfect matching?

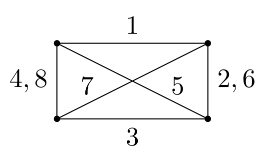
Question 1.5. Is the analysis of Lemma 1.1 tight? It turns out that in practice, the number of iterations performed is much lower than $O(|V|^2)$.

2 The Postman Problem and T-Joins

Think of a postman who wants to deliver mail along every edge of the graph. He wants to visit every edge at least once and return to his starting point.

Definition 2.1. A *postman tour* in a graph is a closed walk traversing each edge at least once.

Example 2.2.

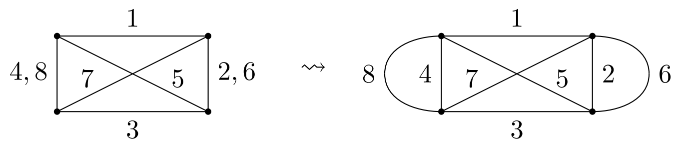


Definition 2.3. An *Eulerian tour* is a closed walk traversing each edge exactly once.

Problem: Given $G = (V, E)$, $c : E \to \mathbb{R}_+$, find a minimum cost postman tour.

Theorem 2.4. *A connected graph G has an Eulerian tour iff every node has even degree in G.*

It turns out that there is a nice correspondence between postman tours and Eulerian tours. Let the sequence $v_1 e_1 v_2 e_2 \cdots v_{n-1} e_{n-1} v_n (= v_1)$ be a postman tour. Duplicate each edge e of G by the number of times, say x_e, it appears in the tour. The resulting graph G^x is an Eulerian tour. Conversely, let $x \in \mathbb{Z}_+^E$ such that G^x is Eulerian. Then x induces a postman tour in G with cost $\sum_{e \in E} c_e x_e$.
Example 2.5.



A minimum cost postman tour x duplicates each edge at most once.

This motivates the following:
Definition 2.6. Given $G = (V, E)$, $T \subseteq V$, a *T-join* of G is a set J of edges of G such that the odd degree nodes of the subgraph (V, J) are exactly T.

The Minimum Cost T-join problem: Given $G = (V, E)$, $c : E \to \mathbb{R}_+$, $T \subseteq V$, the goal is to find a minimum cost T-join.

So to solve the minimum cost postman tour problem, we find a minimum cost T-join J where T is the set of odd-degree of vertices, and then duplicate the edges J, find an Eulerian tour in the augmented graph, and then project back down.

Note that the number of odd degree nodes in a graph is always even.

Before we find a minimum cost T-join, we have to know if there even exists a T-join.
Lemma 2.7. *Let $G = (V, E)$ be connected, and $T \subseteq V$. Then G has a T-join iff $|T|$ is even.*

Proof.

($\Longrightarrow$): Let J be a T-join. The number of odd degree nodes in (V, J) is even. So $|T|$ is even.

($\Longleftarrow$): Let $|T|$ be even, and say $T = \{v_1, v_2, \ldots, v_{2k}\}$. Let P_i be a path from v_i to v_{i+1}. Take H be the multigraph obtained by $\bigcup_{i=1}^{k} P_i$. $\deg_H(v_i)$ is odd for $i = 1, \ldots, 2k$ and every other vertex has even degree. Let e occur $x(e)$ times in H, and $x'(e) = x(e) \bmod 2$. Then $G^{x'}$ is a T-join. ☐

T-joins have very nice properties similar to matchings. If one takes the symmetric difference of a matching with a path, it is still a matching. Something very similar is true for T-joins:
Lemma 2.8. *Suppose $T, T' \subseteq V$, $|T| - |T'|$ even, J' be a T' join. Then J is a T-join iff $J \triangle J'$ is a $(T \triangle T')$-join.*

A sanity check: why should $|T \triangle T'|$ be even? Well if $|T \cap T'|$ is even, then $|T \setminus T'|$ and $|T' \setminus T|$ are even, so $|T \triangle T'| = |(T \setminus T') \cup (T' \setminus T)|$ is even. The case where $|T \cap T'|$ is odd is similar.
Observation 2.9. *$|A \triangle B|$ is even iff $|A|$ and $|B|$ have the same parity.*

Proof of Lemma 2.8.

($\Longrightarrow$): Let $v \in V$. $|(J \triangle J') \cap \delta(v)|$ is even iff $|J \cap \delta(v)|$ and $|J' \cap \delta(v)|$ have the same parity. If both are odd, then $v \in T \cap T'$; otherwise $v \in \overline{T \cup T'}$. Equivalently, $v \notin T \triangle T'$. So $J \triangle J'$ is a $(T \triangle T')$-join.

($\Longleftarrow$): Set $K = J \triangle J'$ and $S = T \triangle T'$. By ($\Longrightarrow$), $(K \triangle J')$ is a $(S \triangle T')$-join, that is, J is a T-join. ☐

Definition 2.10. A *T-path* is a path with its end vertices in T.
Lemma 2.11. *Every T-join is an edge-disjoint union of T-paths and cycles and vice versa.*

Proof.

($\Longleftarrow$) is easy.

($\Longrightarrow$): Assume otherwise. Pick a counterexample T, J with smallest $|T| + |J|$.

Claim 2.12. (V, J) *is a forest.*

Proof. If there exists a cycle C in (V, J), then $J - E(C)$ is still a T-join, so by minimality of the counterexample, $J - E(C)$ is an edge disjoint union of T-paths and cycles, so J is an edge disjoint union of T-paths and cycles as well, contradicting the hypothesis. ■

Pick $u, v \in T$. Fix a T-path P in (V, J) with end vertices u and v. So P is a $\{u, v\}$ join, so $P \triangle J$ is a $(T \triangle \{u, v\})$-join. But $P \triangle J = J - P$ and $T \triangle \{u, v\} = T - \{u, v\}$, so by minimality of the counterexample, $J - P$ is an edge disjoint union of $(T - \{u, v\})$-paths and cycles, so J is an edge disjoint union of T-paths and cycles. □

Lecture 16, October 15, 2015

1 T-Joins, continued

Recall the following.
Definition 1.1. Given $G = (V, E)$, $T \subseteq V$, A *T-join* is a subset J of edges such that the odd degree nodes in the subgrah (V, J) are T.

Goal: Given $G = (V, E)$, $c : E \to \mathbb{R}_{\geqslant 0}$, find a minimum cost T-join.
Lemma 1.2. *G has a T-join iff every connected component of G has an even number of T-vertices.*
Lemma 1.3. *Let $T, T' \subseteq V$, $|T|, |T'|$ even, J' be a T'-join. J is a T-join iff $J \triangle J'$ is a $(T \triangle T')$-join.*
Definition 1.4. A *T-path* is a path with its end vertices in T
Lemma 1.5. *Every T-join is an edge disjoint union of T-paths and cycles and vice versa.*
Claim 1.6. *Any forest with X =the set of odd degree nodes, can be partitioned into edge disjoint paths $P_1, \ldots, P_{|X|/2}$ with pairwise disjoint ends in X.*

Proof. Exercise. □

Corollary 1.7. *Every inclusionwise minimal T-join is an edge disjoint union of T-paths $P_1, \ldots, P_{|T|/2}$ with pairwise disjoint ends in T.*
Observation 1.8. *If $c \geqslant 0$, then there exists a minimum cost T-join that is also inclusionwise minimal.*
Lemma 1.9. *Let $c \geqslant 0$. Then there exists a minimum cost T-join that is the union of $|T|/2$ edge-disjoint shortest paths (with respect to c) joining the nodes of T in pairs.*

Proof. Let J be a minimum cost T-join. Then by the corollary and the observation, $J = \bigcup_{i=1}^{|T|/2} P_i$ where $P_1, \ldots, P_{|T|/2}$ are edge-disjoint T-paths. Let P be one of these paths, say between u and v. Say P' is the shortest path between u and v in G. $J \triangle P \triangle P'$ is a T-join. Then

$$c(J \triangle P \triangle P') = c(J - P) + c(P') - 2c((J - P) \cap P') \leqslant c(J - P) + c(P') = c(J) - (c(P) - c(P')) < c(J),$$

contradicting that J is a minimum cost T-join. □

Algorithm [Edmonds]:

(1) Let $c'(uv)$ = length of shortest uv path with respect to c, and P_{uv} be this path, $c' : E' \to \mathbb{R}_{\geqslant} 0$. Set $H = (T, E')$ where $E' = \{uv : u \in T, v \in T\}$.

(2) Find a minimum c'-cost perfect matching M in H.

(3) Output $\triangle_{uv \in M} P_{uv}$.

Hence Minimum Cost T-join is solvable in polynomial time.

Note that this algorithm only works if $c \geqslant 0$. If costs are negative, then it is still possible, but slightly tricky.

2 T-joins and T-odd cuts

Recall:
Definition 2.1. $U \subseteq V$ is a *T-odd cut* if $U \cap T$ and $\overline{U} \cap \overline{T}$ are odd.

For notational convenience, we will refer to T-odd cuts as *T-cuts*.

T-joins and T-cuts are closely related in a min-max sense.
Lemma 2.2. *Every T-join intersects every T-cut.*

Proof. Let J be a T-join and U be a T-cut. For the sake of contradiction, say $J \cap \delta(U) = \varnothing$. Consider the subgraph $H = (U, J \cap E[U])$. By definition, the odd-degree vertices are those in $T \cap U$, and by assumption, there are an even number of these. But $|U \cap T|$ is odd, and $v \in U \cap T \implies \deg_H(v)$ is odd and $v \in U \setminus T \implies \deg_H(v)$ is even. That is, there are an odd number of odd degree vertices in H, a contradiction. $\square$

Definition 2.3.

1. $P_{T\text{-join}}(G) :=$ convex hull of indicator vectors of T-joins in G.
2. $Q(G) = \{x \in \mathbb{R}^E : x_e \geqslant 0 \ \forall e \in E, x(\delta(U)) \geqslant 1 \ \forall T\text{-cut } U\}$

Theorem 2.4 (Edmonds-Johnson (1973))**.** $Q(G) = P_{T\text{-}join}(G) + \mathbb{R}^E_+$.
Theorem 2.5 (Edmonds-Johnson (1973))**.** *If $c \geqslant 0$, then* $\min\left\{\sum_{e \in E} c_e x_e : x \in Q(G)\right\}$ *gives a minimum cost T-join.*

Suppose we have T-cuts $U_1, \ldots, U_k$ such that $\delta(U_i) \cap \delta(U_j) = \varnothing \ \forall i, j \in [k]$. This means that the minimum cardinality of a T-join is at least k. In particular, the maximum number of edge-disjoint T-cuts is at most the minimum cardinality of a T-join.

This almost gives a min-max relation. It would be very good if the other side were also true. It turns out that it is true in bipartite graphs.
Theorem 2.6 (Seymour (1981))**.** *Let $G = (V, E)$ be bipartite, $T \subseteq V$, $|T|$ even. Then the maximum number of edge-disjoint T-cuts is equal to the minimum cardinality of a T-join.*

Proof (Sebo (1989)). By induction on $|T| + k$. Let J be a minimum cardinality T-join. So (V, J) is a forest, so $\exists v \in T$ such that $\deg_J(v) = 1$. Fix such a vertex v.

We want to try and contract $v \cup N(v)$ to v_N. Define

$$T' := \begin{cases} T - (v \cup N(v)) & \text{if } |(v \cup N(v)) \cap T| \text{ is even} \\ (T - (v \cup N(v))) \cup v_N & \text{otherwise} \end{cases}.$$

Let $e \in J$ be adjacent to v.

Claim 2.7. *$J - e$ is a T'-join in $G' := G/(v \cup N(v))$.*

Proof. Look at $(V(G'), J - e)$ vs (V, J). Observe that $\deg(v_N) = \sum_{u \in v \cup N(v)} \deg(u) - 2$. So $\deg(v_N)$ is even iff there exists an even number of T-vertices in $v \cup N(v)$, i.e. $\deg(v_N)$ is even iff $v_N \notin T'$. ■

G' is bipartite (since parity of cycles do not change).

Claim 2.8. *If $J - e$ is a minimum cardinality T'-join in G', then we are done.*

Proof. Induction tells us that the maximum number of edge-disjoint T'-cuts is equal to the cardinality of $J - e$, which is $|J| - 1$. Let $U_1, \ldots, U_{|J|-1}$ be these cuts. Without loss of generality, we may assume that $v_N \notin U_i \ \forall i = 1, \ldots, |J| - 1$ since otherwise we can just take $\overline{U_i}$ instead. Add a new cut $U_{|J|} = \{v\}$. $\delta(v)$ is disjoint from $\delta(U_1), \ldots, \delta(U_{|J|-1})$, so the maximum number of disjoint T-cuts is at least $|J| - 1 + 1 = |J|$. ■

It turns out that it is not true for all such v that $J - e$ is a minimum cardinality T'-join. So now the goal is to show that there is at least one such v for which $J - e$ has minimal cardinality.

Define $w : E(G) \to \{\pm 1\}$, $w(e) = \begin{cases} -1 & \text{if } e \in J \\ 1 & \text{if } e \in E - J \end{cases}$.

Claim 2.9. *For all cycles C in G, $w(C) \geqslant 0$.*

Proof. Say $w(C) < 0$. Then $C \triangle J$ is a T-join, and

$$|C \triangle J| = |J| - |C \cap J| + |C - J| = |J| + w(C \cap J) + w(C - J) = |J| + w(C) < |J|$$

contradicting J is a minimum cardinality T-join. ■

Let P be a walk without repeating edges of minimal w-weight. Choose P such that P has a minimal number of edges. We will show that one end of P is the vertex that we want.

Claim 2.10. *P is a path.*

Proof. Every cycle has non-negative weight. ■

Let t be an end vertex of P, and f be edge of P incident to t. Note that $w(f) = -1$, so $f \in J$. Furthermore, all the other edges adjacent to t have weight $+1$. So $\deg_J(t) = 1$.

Claim 2.11. *Every cycle C containing t but not f has $w(C) > 0$.*

Proof. If C does not intersect P, then the weight of one edge of C incident to t is $+1$, so the weight of the rest of C is -1, so we can add the rest to P, contradiction.
So C intersects P, say at v, then note that $w(P_{vt}) < 0$, so if Q_1 is a path between v and t in C, then $w(Q_1) > 0$. Similarly, if Q_2 is the other path between v and t in C, then $w(Q_2) > 0$ as well. So $w(C) > 0$. ■

Obtain T', $J' = J - f$.

Claim 2.12. *J' is a minimum cardinality T'-join.*

Proof. If J' is not of minimum cardinality, then there exists a cycle C' in G' such that $|C' - J'| < |C' \cap J'|$. The interesting case is when C' hits t_N. Then blowing t_N back up, we obtain C in G that hits t. have two cases.
The first case is when C does not use f. Then $w(C) > 0 \implies w(C) \geqslant 2$.
[We stopped here. We will continue next lecture.] ■

□

Lecture 17, October 20, 2015

1 Finishing up from last time

We were proving the following theorem:

Theorem 1.1 (Seymour (1981))**.** *Let $G = (V, E)$ be bipartite, $T \subseteq V$, $|T|$ even. Then the maximum number of edge-disjoint T-cuts is equal to the minimum cardinality of a T-join.*

Proof (Sebo (1989)). By induction on $|T| + k$. Let J be a minimum cardinality T-join. So (V, J) is a forest, so $\exists v \in T$ such that $\deg_J(v) = 1$. Fix such a vertex v.

We want to try and contract $v \cup N(v)$ to v_N. Define

$$T' := \begin{cases} T - (v \cup N(v)) & \text{if } |(v \cup N(v)) \cap T| \text{ is even} \\ (T - (v \cup N(v))) \cup v_N & \text{otherwise} \end{cases}.$$

Let $e \in J$ be adjacent to v.

Claim 1.2. *$J - e$ is a T'-join in $G' := G/(v \cup N(v))$.*

G' is bipartite (since parity of cycles do not change).

Claim 1.3. *If $J - e$ is a minimum cardinality T'-join in G', then we are done.*

Define $w : E(G) \to \{\pm 1\}$, $w(e) = \begin{cases} -1 & \text{if } e \in J \\ 1 & \text{if } e \in E - J \end{cases}$.

Claim 1.4. *For all cycles C in G, $w(C) \geqslant 0$.*

Let P be a walk without repeating edges of minimal w-weight. Choose P such that P has a minimal number of edges. We will show that one end of P is the vertex that we want.

Claim 1.5. *P is a path.*

Let t be an end vertex of P, and f be edge of P incident to t. Note that $w(f) = -1$, so $f \in J$. Furthermore, all the other edges adjacent to t have weight $+1$. So $\deg_J(t) = 1$ and $J \cap delta(t) = \{f\}$.

Claim 1.6. *Every cycle C containing t but not f has $w(C) > 0$.*

Obtain T' as before, $J' = J - f$.

Claim 1.7. *J' is a minimum cardinality T'-join.*

Proof. If J' is not of minimum cardinality, then there exists a cycle C' in G' such that $|C'-J'| < |C' \cap J'|$. The interesting case is when C' hits t_N. Then blowing t_N back up, we obtain C in G that hits t. We have two cases.
The first case is when C does not use f. Then $w(C) > 0$. But a cycle has even weight, so $w(C) \geqslant 2$. But $w(C) = |C-J| - |C \cap J|$, so $|C-J| \geqslant |C \cap J| + 2$. Then $|C'-J'| = |C-J| - 2 \geqslant |C \cap J| = |C' \cap J'|$, a contradiction.
So C uses f. Then $|C-J| = |C'-J'| + 1$, and $|C \cap J| = |C' \cap J'| + 1$. Thus $|C-J| < |C \cap J|$, and so $|C \triangle J| \leqslant |J|$, contradicting that J is a minimum T-join. ■

This completes the proof. □

Say $G = K_n$, n even, and $T = V(G)$. A minimum T-join has cardinality $\frac{n}{2}$, but the graph does not even have two edge-disjoint T-cuts, much less $\frac{n}{2}$. So the theorem is false in non-bipartite graphs.

However a nice result still holds in nonbipartite graphs.
Corollary 1.8. *Let $G = (V, E)$, $T \subseteq V$, $|T|$ even. Then*

$$\min_{J : J \text{ is a } T\text{-join}} |J| = \frac{1}{2} \max \{k : \exists\ T\text{-cuts } U_1, \ldots, U_k \text{ such that } \forall e \in E, e \text{ is in at most two sets } \delta(U_i), \delta(U_j)\}$$

Proof. Exercise. □

2 Matroids

Definition 2.1. A *matroid* $\mathcal{M}$ is a pair $(S, \mathcal{I})$ where S is a finite set and $\mathcal{I} \subseteq 2^S$ that satisfy the following:

1. $\varnothing \in \mathcal{I}$
2. $A \in \mathcal{I}, B \subseteq A \implies B \in \mathcal{I}$
3. $A, B \in \mathcal{I}$, $|A| < |B| \implies \exists b \in B \setminus A$ such that $A + b \in \mathcal{I}$.

$\mathcal{I}$ is known as the *independent sets* of the matroid.
Example 2.2. Let S be a set of vectors in $\mathbb{R}^n$, say $a_1, \ldots, a_m$. Let $\mathcal{I}$ be subsets of linearly independent vectors in S. This is known as a *linear matroid*.
Example 2.3. Take $S = \{1, \ldots, n\}$. Fix $k \in \mathbb{N}$, and let $\mathcal{I}$ be the subsets of S of cardinality $\leqslant k$. This is known as a *uniform matroid*.
Example 2.4. Let $G = (V, E)$, take $S = E$ and $\mathcal{I}$ to be subsets of E with no cycles. This is known as a *graphic matroid*.
Definition 2.5. A *base* of a matroid is an inclusionwise maximal subset in $\mathcal{I}$.

Bases have nice properties.
Lemma 2.6. *Let $\mathcal{B}$ be a set of bases of $(S, \mathcal{I})$. Then*

1. *$|B_1| = |B_2|$ for all $B_1, B_2 \in \mathcal{B}$.*
2. *For all $B_1, B_2 \in \mathcal{B}$, $B_1 \neq B_2$, and for all $x \in B_1 \subseteq B_2$, there exists some $y \in B_2 \setminus B_1$ such that $B_1 - x + y \in \mathcal{B}$.*

Proof. (1): Exercise.

(2): Let $A = B_1 - x$, then $|A| < |B_2|$. By the third axiom, $\exists y \in B_2 \setminus A$ such that $A + y \in \mathcal{I}$. But $|A + y| = |B_2|$ so $B_1 - x + y \in \mathcal{B}$. □

Observation 2.7. *Suppose $A \subseteq S$, $A \in \mathcal{I}$. Then there exists a base B containing A.*

This leads us to an alternative definition of a matroid.
Lemma 2.8. *Let S be a finite set, and $\mathcal{B} \subseteq 2^S$ satisfying (1) and (2) of Lemma 2.6. Let*

$$\mathcal{I} = \{U : U \subseteq B \text{ for some } B \in \mathcal{B}\}.$$

Then $(S, \mathcal{I})$ is a matroid.

Proof. Exercise. □

Problem: Given a matroid $(S, \mathcal{I})$, weights $w \colon S \to \mathbb{R}$, find an $I \in \mathcal{I}$ with maximum $w(I) = \sum_{e \in I} w_e$.

This captures a large family of problems.

Note that $\mathcal{I}$ could have size exponential in the size of S. So typically, the matroid $(S, \mathcal{I})$ is given by an *independence-testing oracle*, i.e., given $T \subseteq S$, the oracle answers whether $T \in \mathcal{I}$. This is valid because in most problems, testing whether or not $T \in \mathcal{I}$ can be done efficiently.

Greedy Algorithm
1 Order elements of S as $e_1, \ldots, e_m$ such that $w(e_1) \geqslant w(e_{i+1})$ for all i
2 Start with $I = \varnothing$. Try and add elements one at a time, maintaining $I \in \mathcal{I}$.

To show the correctness, we need the following:
Definition 2.9. Let $(S, \mathcal{I})$ be a matroid, $T \subseteq S$.

1. The *rank* of T is $r(T) := \max_{I \in \mathcal{I}, I \subseteq T} |I|$.
2. A *base of* T is an inclusionwise maximal independent set in T.

It is clear that $r(T)$ is the size of a base of T.
Example 2.10. In the case of the graphic matroid, $r(T) = n - \#$connected components in (V, T).

Let us formulate the problem as an IP:

$$\begin{aligned} \max \sum_{e \in S} w_e x_e & \\ \sum_{e \in T} x_e \leqslant r(T) & \qquad \forall T \subseteq S \\ x \in \{0,1\}^S & \end{aligned}$$

Exercise. $J \in \mathcal{I} \iff \sum_{e \in T} \chi_J(e) \leqslant r(T) \; \forall T \subseteq S$.

So for the polyhedron $P = \left\{x \in \mathbb{R}^S : x(T) \leqslant r(T) \forall T \subseteq S, x \geqslant 0\right\}$, $P \subseteq \mathbb{Z}^S$ are the independent sets.

The LP relaxation is $\max\left\{\sum_{e \in E} w_e x_e : \sum_{e \in T} x_e \leqslant r(T) \; \forall T \subseteq S, x \geqslant 0\right\}$. We will argue that the Greedy Algorithm finds an optimal solution to the LP, then since the solution is integral, it is a solution to the IP.
Lemma 2.11. *$\forall w \colon S \to \mathbb{R}$, the optimal solution to the LP-relaxation is obtained by the indicator vector of the set J constructed by the Greedy Algorithm.*

Proof. Without loss of generality assume $w \colon S \to \mathbb{R}_{\geqslant 0}$. Look at the dual:

$$\begin{aligned} \min \sum_{T \subseteq S} r(T) y_T & \\ \sum_{T \subseteq S : e \in T} y_T \geqslant w_e & \qquad \forall e \in S \\ y \geqslant 0 & \end{aligned}$$

Assume $e_1, \ldots, e_m$ are ordered so that $w(e_i) \geqslant w(e_{i+1})$.

Let J be the set constructed by the Greedy Algorithm, and $\overline{X}$ be the indicator vector of J, $T_0 = \varnothing$, $T_i = \{e_1, \ldots, e_i\}$. Define $\overline{Y}$ by

$$\overline{Y} = \begin{cases} w(e_i) - w(e_{i+1}) & \forall T = T_i, i = 1, \ldots, m-1 \\ w(e_m) & \text{if } T = T_m \\ 0 & \text{otherwise} \end{cases}$$

Fix $e_i \in S$. Then

$$\sum_{T \subseteq S : e_i \in T} \overline{Y}_T = \sum_{j=i}^{n} \overline{Y}_{T_j} = (w(e_i) - w(e_{i+1})) + (w(e_{i+1}) - w(e_{i+2})) + \cdots + w(e_m) = w(e_i).$$

So $\overline{Y}$ is dual feasible.

If $\overline{X}_e > 0$, then $\sum_{T \subseteq S : e \in T} \overline{Y}_T = w_e$, and if $\overline{Y}_T > 0$, $\overline{X}(T) = r(T)$.

[We stopped here. We will continue next lecture.] $\square$

Lecture 18, October 22, 2015

1 Finishing up from last time

Recall:

Problem: Given a matroid $(S, \mathcal{I})$ (by an independence testing oracle), weights $w\colon S \to \mathbb{R}$, find an $I \in \mathcal{I}$ with maximum $w(I) = \sum_{e \in I} w_e$.

We had an LP relaxation:

$$\begin{aligned} \max \sum_{e \in S} w_e x_e & \\ \sum_{e \in T} x_e \leqslant r(T) & \qquad \forall T \subseteq S \\ x_e \geqslant 0 & \qquad \forall e \in S \end{aligned}$$

and the dual:

$$\begin{aligned} \min \sum_{T \subseteq S} r(T) y_T & \\ \sum_{T \subseteq S : e \in T} y_T \geqslant w_e & \qquad \forall e \in S \\ y \geqslant 0 & \end{aligned}$$

We were proving the following lemma:
Lemma 1.1. *$\forall w\colon S \to \mathbb{R}$, the optimal solution to the LP-relaxation is obtained by the indicator vector of the set J constructed by the Greedy Algorithm.*

Proof. Without loss of generality assume $w\colon S \to \mathbb{R}_{\geqslant 0}$. Look at the dual:

Assume $e_1, \ldots, e_m$ are ordered so that $w(e_i) \geqslant w(e_{i+1})$.

Let J be the set constructed by the Greedy Algorithm, and $\overline{X}$ be the indicator vector of J. Let $T_0 = \varnothing$, $T_i = \{e_1, \ldots, e_i\}$. Define $\overline{Y}$ by

$$\overline{Y} = \begin{cases} w(e_i) - w(e_{i+1}) & \forall T = T_i, i = 1, \ldots, m-1 \\ w(e_m) & \text{if } T = T_m \\ 0 & \text{otherwise} \end{cases}$$

$\overline{Y}$ is dual feasible.

If $\overline{X}_e > 0$, then $\sum_{T \subseteq S : e \in T} \overline{Y}_T = w_e$.

If $\overline{Y}_T > 0$, $\overline{X}(T) = r(T)$: Note that $\overline{Y}_T > 0 \iff T = T_i$ for some i. So we need to verify $|J \cap T_i| = r(T_i)$ for $i = 1, \ldots, m$, i.e., verify that $J \cap T_i$ is a base of T_i for $i = 1, \ldots, m$. Assume not. Pick a minimal i such that $J \cap T_i$ is not a base of T_i. Hence $U = J \cap T_{i-1}$ is a base for T_{i-1}. By the choice of i, U cannot be a base of T_i. So say U' is a base of T_i. We know that $|U'| \geqslant |U|$; if $|U'| = |U|$ then U is a base of T_i, a contradiction. So $|U'| = |U| + 1$, and $e_i \in U'$ (for otherwise, $e_i \notin U'$, so $U' \subseteq T_{i-1}$, contradicting that U is a base of T_{i-1}). So there exists some $z \in U' \setminus U$ such that $U + z \in \mathcal{I}$. If $z \neq e_i$, then $U + z \subseteq T_{i-1}$, a contradiction. So $z = e_i$, then $(J \cap T_{i-1}) + e_i \in \mathcal{I}$, so Greedy would have picked e_i and added it to J, so $J \cap T_i$ is a base of T_i. Contradiction. □

Some consequences:

Theorem 1.2. *Let $(S, \mathcal{I})$ be a matroid. Then the polyhedron $\{x \in \mathbb{R}^S : x(T) \leqslant r(T)\ \forall T \subseteq S, x \geqslant 0\}$ is an integral polyhedron and the convex hull of incidence vectors of sets in $\mathcal{I}$. The polyhedron is known as the* matroid polyhedron *or the* independence polyhedron.

Theorem 1.3. *The system $x(T) \leqslant r(T)\ \forall T \subseteq S, x_e \geqslant 0\ \forall e \in S$ is TDI.*

Theorem 1.4. *There exists a dual optimum supported on a nested family of subsets of S, which is a special kind of laminar.*

Minimum spanning tree can be modeled as finding a maximum independent set in the graphic matroid. Then Kruskal's algorithm is exactly the Greedy algorithm we described.

In fact, the Greedy algorithm characterizes the matroid structure:

Theorem 1.5. *Let S be a finite set, $\mathcal{I} \subseteq 2^S$. Say $(S, \mathcal{I})$ satisfy axioms (1) and (2). $(S, \mathcal{I})$ is a matroid iff the Greedy algorithm obtains a maximum w-weight set in $\mathcal{I}$ for all $w \colon S \to \mathbb{R}$.*

Proof. We have already seen the ($\Longrightarrow$) direction.

($\Longleftarrow$): Suppose not. Say $A, B \in \mathcal{I}$, $|A| < |B|$, and $\forall e \in B \setminus A$, $A + e \notin \mathcal{I}$. We will show that Greedy will fail for a particular weight function. Define w as follows: for $e \in \overline{A \cup B}$, $w(e) = 0$, for $e \in A$, $w(e) = 1$, and finally for $e \in B \setminus A$, $w(e) = \frac{|A-B|}{|B-A|} + \varepsilon$ for some $\varepsilon > 0$ such that $\frac{|A-B|}{|B-A|} + \varepsilon < 1$. By hypothesis, $w(J_{\text{Greedy}}) = w(A) = |A|$. But $w(B) = |B \cap A| + |B - A| \left(\frac{|A-B|}{|B-A|} + \varepsilon \right) = |A| + |B - A| \cdot \varepsilon > w(A) = w(J_{\text{Greedy}})$. Hence Greedy is not optimal. □

2 Matroid Representation

We saw that a matroid can be represented by a basis and a rank function oracle. Next we will define the circuit oracle.

Recall that a base is an inclusionwise maximal independent set in the matroid.

Definition 2.1. A *circuit* C in a matroid is an inclusionwise minimal dependent set. That is, $\forall e \in C$, $C - e \in \mathcal{I}$.

Example 2.2. In graphic matroids, circuits correspond exactly to cycles.

Lemma 2.3. *Let C and C' be circuits of a matroid $(S, \mathcal{I})$, with $e \in C \cap C'$, $f \in C \setminus C'$. Then there exists a circuit C'' containing f but not e.*

Proof. $C - f$ and $C' - e$ are independent, so there exists a base $B \supseteq C - f$ of $C \cup C'$, $f \notin B$, and a base $B' \supseteq C' - e$ of $C \cup C'$, $e \notin B'$.

Suppose $f \notin B'$. Then $B' + f \notin \mathcal{I}$, so it should contain a circuit, say C'', with $f \in C''$ and $e \notin C''$ and we are done.

Now suppose $f \in B'$. Let us use the base exchange property: $f \in B' \setminus B$, so there exists $z \in B \setminus B'$ such that $B'' = B' - f + z$ is a base of $C \cup C'$. If $e = z$, then B'' contains C', a contradiction. So $e \notin B''$, then B'' contains $C' - e$ and $f \notin B''$. Apply the previous case. □

Exercise. $\forall B \in \text{Base}(S, \mathcal{I})$, $\forall e \in S - B$, $B + e$ has a unique circuit.

Lemma 2.4. *Let S be a finite set, $\mathcal{C} \subseteq 2^S$ such that $\forall C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$,*

1. *$C_i \nsubseteq C_2$ and $C_2 \nsubseteq C_1$,*
2. *$\forall e \in C_1 \cap C_2$, $\exists C' \in \mathcal{C}$ such that $C' \subseteq (C_1 \cup C_2) - e$.*

Let $\mathcal{I} = \{A \in 2^S : \text{no } C \in \mathcal{C} \text{ is a subset of } A\}$. Then $(S, \mathcal{I})$ is a matroid.

Proof. Exercise. □

Definition 2.5. Given $A \subseteq S$, a *circuit oracle* responds YES iff A is a circuit in the matroid.

So given a circuit oracle, a natural question is, can we design an independence oracle that makes polynomially many calls to the circuit oracle. It turns out that this is not possible.

Lemma 2.6. *Given a circuit oracle for a matroid $(S, \mathcal{I})$, there does not exist a* $\text{poly}(|S|)$*-time algorithm to determine if a given set is independent.*

In order to prove this, we need the following definitions:

Definition 2.7. A *loop* of a matroid is an element e such that $\{e\}$ is a circuit. A *coloop* is an element not present in any circuit.

Example 2.8. In a graphic matroid, coloops are cut edges.

Proof of Lemma 2.6. Let $S = [n]$. For each subset $U \subseteq [n]$ of size $\frac{n}{2}$, define $\mathcal{M}_U$ as the matroid with one circuit equal to U, and all the rest are coloops. This defines an exponential family of matroids. Also define $\mathcal{M}$ as the matroid where all elements are coloops.

So given a circuit oracle of a matroid $\mathcal{N}$, we want to determine if $T = [n]$ is independent in $\mathcal{N}$. For any queried subset $A \subseteq [n]$, the oracle will return NOT A CIRCUIT. Suppose the number of queries made by the algorithm is polynomial, then for each such set of queries $\{A_1, \ldots, A_{\text{poly}(n)}\}$, there exists a subset $U \subseteq [n]$, $|U| = \frac{n}{2}$ which has $U \neq A_i \; \forall i = 1, \ldots, \text{poly}(n)$. Hence the circuit oracle for $\mathcal{M}_U$ and $\mathcal{M}$ respond identically for all A_i's queried. So the algorithm cannot distinguish whether the circuit oracle is for $\mathcal{M}$ or $\mathcal{M}_U$, and therefore cannot determine if $T = [n]$ is independent or not. □

Let us return to the matroid polyhedron $\{x \in \mathbb{R}^S : x(T) \leqslant r(T) \; \forall T \subseteq S, x \geqslant 0\}$. Note that there are exponentially many constraints. So the question is if we need all of these constraints. So we will derive a minimal representation of the polyhedron.

Definition 2.9. A function $f \colon 2^S \to \mathbb{R}$ is *submodular* if $\forall A, B \subseteq S$, $f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B)$.

Lemma 2.10. *The rank function of a matroid is submodular.*

Lecture 19, October 27, 2015

1 Representations of the Matroid Polyhedron

Recall the matroid polyhedron $\{x \in \mathbb{R}^S : x(T) \leqslant r(T)\ \forall T \subseteq S, x \geqslant 0\}$. Note that there are exponentially many constraints. So the question is if we need all of these constraints. So we will derive a minimal representation of the polyhedron.

Definition 1.1. A function $f\colon 2^S \to \mathbb{R}$ is *submodular* if $\forall A, B \subseteq S$, $f(A) + f(B) \geqslant f(A \cup B) + f(A \cap B)$.

Recall the rank function: $r(U)$ is the size of a maximum independent set in U.

Lemma 1.2. *The rank function of a matroid is submodular.*

Proof. Fix $A, B \subseteq S$. Let J be a base of $A \cap B$, and extend J to a base I of $A \cup B$.

$r(A) \geqslant |I \cap A|$, $r(B) \geqslant |I \cap B|$. Then

$$r(A) + r(B) \geqslant |I \cap A| + |I \cap B| = |I \cap (A \cap B)| + |I \cap (A \cup B)| = |J| + |I| = r(A \cap B) + r(A \cup B). \quad \square$$

Next, we need the notion of a flat of a matroid.

Definition 1.3. $F \subseteq S$ is a *flat* if $\forall e \in S - F$, $r(F + e) > r(F)$.

Example 1.4. In graphic matroids, flats correspond to vertex-induced subgraphs (recall the rank function is $n -$ #connected components; every edge outside F connects two different connected components of (V, F) $\iff$ F contains all edges of G connecting any two end vertices of $V(F)$).

Proposition 1.5. *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid, $U \subseteq S$. Then there exists a flat $U' \supseteq U$ where $r(U') = r(U)$.*

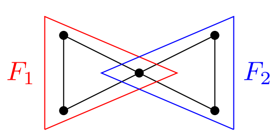
Proof. We can keep adding to U without changing the rank; when adding another element would increase the rank, we have a flat. $\square$

Exercise. Show that U' is unique.

Let $A \subseteq S$. For any partition $A = F_1 \sqcup F_2$, $r(A) \leqslant r(F_1) + r(F_2)$.

Definition 1.6. A set $U \subseteq S$ is *inseparable* if there does not exist a partition $F_1, F_2 \neq \varnothing$ of U such that $r(U) = r(F_1) + r(F_2)$.

Example 1.7. In a graphic matroid, take for example:



$r(U) = 4$, $r(F_1) = 2 = r(F_2)$, so U is separable. In general, U is separable iff the vertex connectivity of $G[U]$ is at least two.

Proposition 1.8. *Let $\mathcal{M} = (S, \mathcal{I})$ be matroid with no loops. If $U \subseteq S$ is inseparable, then for all $e \in S - U$ such that $r(U + e) = r(U)$, $U + e$ is inseparable.*

Proof. Assume not. Pick a counterexample $(S, \mathcal{I})$ with minimum $|S|$. Then let $S = U + e$, U inseparable, $U + e$ separable, with $r(U + e) = r(U)$. Since $U + e$ is separable, then there exists a partition $F_1, F_2 \neq \varnothing$ of U such that $r(F_1) + r(F_2) = r(U + e) = r(U)$. Without loss of generality, say $e \in F_1$.

Claim 1.9. $F_1 \setminus e \neq \varnothing$.

Proof. Say $F_1 \setminus e = \varnothing$, so $F_1 = \{e\}$, $F_2 = U$. Then

$$r(F_2) = r(U) = r(U+e) = r(F_1) + r(F_2) \implies r(F_1) = 0 \implies r(\{e\}) = 0,$$

so e is a loop, contradiction. ■

Then $F_1 \setminus e, F_2 \neq \varnothing$ is a partition of U. By minimality, $r(U) < r(F_1 \setminus e) + r(F_2) \leqslant r(F_1) + r(F_2) = r(U)$, a contradiction. □

Theorem 1.10. *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid with no loops. Then an inclusionwise minimal system for the matroid polyhedron is given by*

$$\begin{aligned} x(T) &\leqslant r(T) \quad && \forall T \subseteq S \text{ inseparable flats} \\ x_e &\geqslant 0 && \forall e \in S \end{aligned}$$

Proof. Call the polyhedron given by the constraints Q. We first show that Q equals the matroid polyhedron.

($\supseteq$) is easy.

($\subseteq$): Let $x \in Q$. Suppose there exists $U \subseteq S$ such that $x(U) > r(U)$. Pick such U with minimum $r(U)$.

Claim 1.11. *U is inseparable.*

Proof. Say U is separable, then there exists a partition $F_1, F_2 \neq \varnothing$ of U such that $r(U) = r(F_1)+r(F_2)$. Since the matroid has no loops, $r(U) > r(F_1)$, $r(U) > r(F_2)$. By minimality of $r(U)$, $x(F_1) \leqslant r(F_1)$, $x(F_2) \leqslant r(F_2)$, so

$$x(U) = x(F_1) + x(F_2) \leqslant r(F_1) + r(F_2) = r(U)$$

a contradiction. ■

By Propositions 1.5 and 1.8, there exists $U' \subseteq U$ such that U' is an inseparable flat and $r(U') = r(U)$. Then $r(U) < x(U) \leqslant x(U') \leqslant r(U')$, a contradiction.

Next, we show that the system is inclusionwise minimal.

If we allow $x_e < 0$, then let $\bar{x}$ be such that $\bar{x}_f = \begin{cases} 0 & \text{if } f \neq e \\ -1 & \text{if } f = e \end{cases}$. Then $\bar{x}$ satisfies all other inequalities but $\bar{x}$ is not in the matroid polyhedron.

To show that $x(T) \leqslant r(T)$ is necessary for all inseparable flats T, consider the face $R = Q \cap \{x : x(T) = r(T)\}$. If $x(T) \leqslant r(T)$ is not necessary, then R is contained in a facet of Q. Now

$$\begin{aligned} R &= \left\{x \in \mathbb{R}^S : x \in Q, x(T) = r(T)\right\} \\ &\subseteq \left\{x \in \mathbb{R}^S : x(F) = r(F), x_e \geqslant 0\ \forall e \in S, x(U) \leqslant r(U)\ \forall \text{inseparable flats } U \neq T\right\}. \end{aligned}$$

Claim 1.12. *For $F \subseteq S$, $F \neq \varnothing$, F is an inseparable flat, there exists a base I of T such that $|I \cap F| < r(F)$.*

Proof. Assume not. Then for all bases I of T, $|I \cap F| = r(F)$. Let I' be a base of $T - F$, then extend I' to a base of T. By submodularity,

$$r(F) \geqslant r(T \cap F) \geqslant r(T) - r(T - F) = |I| - |I - F| = |I \cap F| = r(F)$$

so $r(T) = r(T \cap F) + r(T - F)$, and since T is inseparable, either $T \cap F$ or $T - F$ is empty.
Case 1: Say $T - F = \varnothing$, so $T \subseteq F$. But T is a flat, so $r(T) < r(F)$, but $r(T) = r(T \cap F) = |I \cap F|$, contradicting $r(F) = |I \cap F|$.
Case 2: Say $T \cap F = \varnothing$, so $|I \cap F| = 0$, and $r(F) = 0$, a contradiction since $F \neq \varnothing$ and we have no loops. ■

Let $\bar{x}$ be the indicator vector of I. $\bar{x}(T) = r(T) \implies \bar{x} \in R$, but $\bar{x}(F) = |I \cap F| < r(F)$. So $R \subsetneq Q$. □

2 Matroid Intersection

Definition 2.1. A matroid $\mathcal{M} = (S, \mathcal{I})$ is a *partition matroid* if there exists a partition $S_1, \dots, S_k$ of the ground set S such that $I \in \mathcal{I} \iff |I \cap S_i| \leqslant 1 \ \forall i \in [k]$.
Example 2.2.

1. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, $S = E$ with partitions $S_v = \delta(v) \ \forall v \in V_1$. A set of edges each of which covers a distinct $v \in V_1$ is an independent set.
2. Let $G = (V, E)$ be a directed graph, $S = E$ with partition $S_v = \delta^{\text{in}}(v) \ \forall v \in V$. A cycle is an independent set.

One can model bipartite matchings using the intersection of two matroids.

Lecture 20, October 29, 2015

1 Matroid Intersection

Definition 1.1. Given two matroids $M_1 = (S, \mathcal{I}_1)$, $M_2 = (S, \mathcal{I}_2)$, the intersection of M_1 and M_2 is the tuple $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$.

The intersection of two matroids need not be a matroid.
Example 1.2. Set $S = \{a, b, c\}$, $\mathcal{I}_1 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$, $\mathcal{I}_2 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$.

Then $\mathcal{I}_1 \cap \mathcal{I}_2 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, c\}\}$, and $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ is not a matroid: $|\{a, c\}| > |\{b\}|$ but there is no element in $\{a, c\}$ that can be added to $\{b\}$ to obtain an independent set. Alternatively, let $w\colon S \to \mathbb{R}$, $w(a) = 2$, $w(b) = 4$, $w(c) = 3$. The greedy solution has weight 4, which does not agree with the optimum solution which has weight 5.

Problem. Maximum weight common independent set

- Given: $\mathcal{M}_i = (S, \mathcal{I}_i)$; $i = 1, 2$; $w\colon S \to \mathbb{R}$
- Goal: Find a set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ of max $w(I)$.

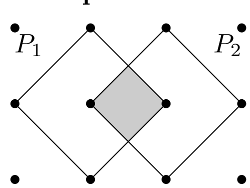
Example 1.3. Given $G = (V_1 \cap V_2, E)$ bipartite, let M_i be the partition matroid $S = E$, partition $(\delta(v))_{v \in V_i}$. Then $I \in \mathcal{I}_1 \cap \mathcal{I}_2 \iff |I \cap \delta(v)| \leqslant 1\ \forall v \in V_1 \cap V_2$, so I is a matching.
Example 1.4. Given $G = (V, E)$ directed, let M_1 be the partition matroid $S = E$, partition $(\delta^{\mathrm{IN}}(v))_{v \in V}$, and M_2 be the graphic matroid $S = E$, $\mathcal{I} = \{F \subseteq E : \text{undirected version of } (V, F) \text{ has no cycles}\}$. Then $I \in \mathcal{I}_1 \cap \mathcal{I}_2 \iff I$ is a *branching*.
Definition 1.5. A *branching* is a directed forest where each vertex has indegree at most 1.
Theorem 1.6 (Edmonds). *The convex-hull P of incidence vectors of common independent sets is given by*

$$Q = \left\{x \in \mathbb{R}^S : x(A) \leqslant r_1(A)\ \forall A \subseteq S, x(A) \leqslant r_2(A)\ \forall A \subseteq S, x_e \geqslant 0\ \forall e \in S\right\}.$$

Q is known as the matroid intersection polytope

Note that Q has exponentially many constraints, but we were able to give a polynomially-sized polyhedral description for bipartite matching. This is because the degree constraints correspond to inseparable flats.

That $P \subseteq Q$ is obvious. But it is not always true that the intersection of two integral polyhedra is integral, indeed, consider the following:
Example 1.7.



To show that $Q \subseteq P$, we need to show that Q is integral.

The approach is as follows: in a system $Ax \leqslant b$, it is not necessary to show that A is TU. Look at an extreme point $\bar{x}$, then $\bar{x}$ satisfies $\bar{A}\bar{x} = \bar{b}$, where $\bar{A}x \leqslant \bar{b}$ is a subsystem. Then if $\bar{A}$ is TU, then $\bar{x}$ is integral.

Let us write down the primal and dual:

$$(\mathrm{P}) : \max \sum_{e\in S} w_e x_e \qquad\qquad (\mathrm{D}) : \min \sum_{A\subseteq S} y^1_A \gamma_1(A) + y^2_A \gamma_2(A)$$

$$x \in Q \qquad\qquad \sum_{A\subseteq S: e\in A} \big(y^1_A \gamma_1(A) + y^2_A \gamma_2(A)\big) \geqslant w_e \quad \forall e \in S$$

$$y^1_A, y^2_A \geqslant 0 \quad \forall A \subseteq S$$

If $(\bar{y}^1, \bar{y}^2)$ is dual optimal, define w^1, w^2 as $w^i_e := \sum_{A\subseteq S: e\in A} \bar{y}^i_A$ for $i = 1, 2$.

Claim 1.8. *$(\bar{z}^1, \bar{z}^2)$ is optimal (D) iff $\bar{z}^i$ is optimal to*

$$(\mathrm{D}^i) : \min \sum_{A\subseteq S} z^i_A \gamma_i(A)$$

$$\sum_{A\subseteq S: e\in S} z^i_A \geqslant w^i_e \quad \forall e \in S$$

$$z^i_A \geqslant 0 \quad \forall A \subseteq S$$

which is the dual of the max w^i-weight independent set LP for $\mathcal{M}_i$.

Proof. ($\Longrightarrow$) : Suppose $\bar{z}^1$ is not optimal to (D^1), then there exists $\bar{\alpha}^1$ optimal to (D^1) and $\sum_{A\subseteq S} \bar{\alpha}^1_A \gamma_1(A) < \sum_{A\subseteq S} \bar{z}^1_A \gamma_1(A)$, then $(\bar{\alpha}^1, \bar{z}^2)$ is feasible for (D), contradicting the optimality of $(\bar{y}^1, \bar{y}^2)$.

($\Longleftarrow$) : Similar. □

Theorem 1.9. *The system defining Q is TDI.*

Proof. Given $w : S \to \mathbb{Z}$, the goal is to show that (D) has an integral optimum.

Choose $(\bar{y}^1, \bar{y}^2)$ to be dual optimal. Choose w^1, w^2 as before. Then $\bar{y}^i$ is optimal to (D^i). We can assume that $\bar{y}^i$ is the dual optimum corresponding to the greedy algorithm that finds the max w^i-weight independent set in $\mathcal{M}_i$, in particular, there exists a sequence $T^i_1 \subseteq T^i_2 \subseteq \cdots T^i_{N_i}$ such that $\bar{y}^i_A > 0 \iff A = T^i_j$, $j \in [N_i]$ for $i = 1, 2$.

This means there is an optimum to (D) which is also optimal to

$$(\mathrm{D}') : \min \sum_{A\in X_1} y^1_A \gamma_1(A) + \sum_{A\in X_2} y^2_A \gamma_2(A)$$

$$\sum_{A\in X_1: e\in A} y^1_A + \sum_{A\in X_2: e\in A} y^2_A \geqslant w_e \quad \forall e \in S$$

$$y^i_A \geqslant 0 \quad \forall A \in X, i = 1, 2$$

where $X_i = \{T^i_j : j \in [N_i]\}$.

We will prove that the constraint matrix to (D') is TU. Consider the transpose of the constraint matrix of (D'), call it B. Each column corresponds to to an element, and each row corresponds to an indicator vector of T^i_j. Without loss of generality, suppose the rows of B are split into two parts, B_2 whose rows are $T^2_{N_2}, T^2_{N_2-1}, \ldots, T^2_1$, and B_1 whose rows are $T^1_{N_1}, T^1_{N_1-1}, \ldots, T^1_1$.

Claim 1.10. *B is TU.*

Proof. Suppose not. Pick a smallest counterexample. Every square submatrix of B has determinant $0, \pm 1$, and $\det(B) \notin \{0, \pm 1\}$. So there is no row or column with $\leqslant 1$ one. So we cannot have two rows R_1 and R_2 in B_i such that the number of ones in R_1 and the number of ones in R_2 differ by at most one.
Without loss of generality, suppose $N_1 \geqslant N_2$, then $N_1 \geqslant \frac{n}{2}$. The smallest row has at least 2 ones, the second smallest row has at least 4 ones, and so on, so B_1 has exactly $\frac{n}{2}$ rows and $T^1_{N_1}$ is a row of all ones. Similarly, $T^1_{N_2}$ is also a row of all ones. Then $\det(B) = 0$, a contradiction. ■

This along with Claim 1.8 show that Q is TDI. □

This does not generalize to the intersection of three matroids.

Some consequences:

- This proves Theorem 1.6.
- There is a dual optimum supported on two nested families.
- The rank constraint matrix for a single matroid is TDI.

Lecture 21, October 30, 2015

Someday... :(

Lecture 22, November 3, 2015

1 Maximum Cardinality Common Independent Set Problem

Recall:

- Given: $\mathcal{M}_i = (S, \mathcal{I}_i)$, $i = 1, 2$
- Goal: Find $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ with maximum $|I|$.

Theorem 1.1.

$$\max_{i \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq S} (r_1(U) + r_2(S - U))$$

Algorithm:

- Given: $I \in \mathcal{I}_1 \cap \mathcal{I}_2$
- Goal: Find $I' \in \mathcal{I}_1 \cap \mathcal{I}_2$, $|I'| > |I|$ or certify optimality of I

1. Find $X_i := \{x \in S - I : I + x \in \mathcal{I}_i\}$, $i = 1, 2$
2. Build directed graph on vertex set S where $y \to x$ if $I + x - y \in \mathcal{I}_1$, $x \to y$ if $I + x - y \in \mathcal{I}_2$
3. If P is a $X_1 \to X_2$ path of minimal length, then $I \triangle V(P) \in \mathcal{I}_1 \cap \mathcal{I}_2$. Return $I \triangle V(P)$

Lemma 1.2. *If there is no directed $X_1 \to X_2$ path, then I is an optimum.*

Proof. Since there is no $X_1 \to X_2$ path, there is a cut $U \subseteq S$ such that $X_1 \subseteq \overline{U}$, $X_2 \subseteq U$ and all edges between U and $\overline{U}$ are from U to $\overline{U}$.

Claim 1.3. $r_1(U) + r_2(S - \overline{U}) = |I|$.

Proof. We will show that $I \cap U$ is a base of U. This is sufficient since $|I \cap U| = r_1(U)$, $|I \cap \overline{U}| = r_2(U)$, so $|I| = |I \cap U| + |I \cap \overline{U}| = r_1(U) + r_2(S - U)$.
Since $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, $I \cap U \in \mathcal{I}_1$. If $I \cap U$ is not a base of U, then there exists $x \in U - I$ such that $I \cap U + x \in \mathcal{I}_1$. But $x \notin X_1$, so $I + x \notin \mathcal{I}_1$, and so there exists a circuit C in $I + x$ of $\mathcal{M}_1$ containing x and some $y \in I \cap \overline{U}$. But then $I + x - y \in \mathcal{I}_1$, which gives an arc $y \to x$, contradiction.
A similar argument shows $|I \cap \overline{U}|$ is a base of $S - U$. ■

Optimality follows from Theorem 1.1. □

We can also find efficient algorithms to solve the Maximum Weight Common Independent Set problem.

2 Submodular Functions

Recall:

Definition 2.1. A set function $f : 2^S \to \mathbb{R}$ is *submodular* if $\forall U, V \subseteq S$, $f(U) + f(V) \geqslant f(U \cap V) + f(U \cup V)$.

Example 2.2.

- The rank function of a matroid is submodular
- The cut function of a graph is submodular
  - $G = (V, E)$ undirected, $c : E \to \mathbb{R}_+$, $f : 2^V \to \mathbb{R}_+$ is $f(U) = \sum_{e \in \delta(U)} c(e)$
  - $G = (V, E)$ directed, $c : E \to \mathbb{R}_+$, $f : 2^V \to \mathbb{R}_+$ is $f(U) = \sum_{e \in \delta^{\text{out}}(U)} c(e)$
- Matroid intersection: Given $\mathcal{M}_i = (S, \mathcal{I}_i)$, $i = 1, 2$, then $f : 2^S \to \mathbb{R}$, $f(U) = r_1(U) + r_2(S - U)$

The problem we are interested in is

$$\min_{U \subseteq S} f(U)$$

which clearly generalizes the min-cut problem, and for matroid intersection, is the problem in Theorem 1.1.

Proposition 2.3. *A set function $f : 2^S \to \mathbb{R}$ is submodular iff $f(U + s) + f(U + t) \geqslant f(U) + f(U + s + t)$ $\forall U \subseteq S$, $s, t \in S$.*

Proof. ($\implies$) is trivial.

($\impliedby$): By induction on $|U \triangle V|$.

When $|U \triangle V| \leqslant 1$, then without loss of generality $U \subseteq V$; the inequality holds trivially.

When $|U \triangle V| = 2$ then either $U \subseteq V$, otherwise $s \in U \setminus V$, $t \in V \setminus U$. Set $U' = U \cap V$. Then $U' + s = U$, $U' + t = V$, $U' + s + t = U \cup V$.

Now let $|U \triangle V| \geqslant 3$, now say $|V - U| \geqslant 2$. Fix $t \in V - U$. Look at $(V - t) \triangle U$, $|(V - t) \triangle U| < |V \triangle U|$. By induction hypothesis,

$$f(V - t) + f(U) \geqslant f((V - t) \cap U) + f((V - t) \cup U)$$

or, written another way,

$$f(U) - f(V \cap U) \geqslant f((V - t) \cup U) - f(V - t).$$

Now look at $((V - t) \cup U) \triangle V = (U - V) \cup \{t\}$, $|(U - V) \cup \{t\}| < |V \triangle U|$. By induction hypothesis,

$$f(V) + f((V - t) \cup U) \geqslant f(((V - t) \cup U) \cap V) + f(((V - t) \cup U) \cup V)$$

and we obtain

$$f((V - t) \cap U) + f((V - t) \cup U) \geqslant f(V \cup U) - f(V)$$

which gives the desired inequality. □

There are two polyhedra associated with a submodular function.

Definition 2.4. The *polymatroid* associated with a submodular function $f : 2^S \to \mathbb{R}$ is

$$P_f = \left\{ x \in \mathbb{R}^S : x_e \geqslant 0 \ \forall e \in S, x(U) \leqslant f(U) \ \forall U \subseteq S \right\}.$$

The *extended polymatroid* associated with f is

$$EP_f = \left\{ x \in \mathbb{R}^S : x(U) \leqslant f(U) \ \forall U \subseteq S \right\}.$$

Observation 2.5. $P_f \neq \varnothing \iff f(U) \geqslant 0 \ \forall U \subseteq S$. $EP_f \neq \varnothing \iff f(\varnothing) \geqslant 0$.

Proposition 2.6. *Suppose $f : 2^S \to \mathbb{R}$ is submodular, $x \in EP_f$. Then the family $\{U \subseteq S : x(U) = f(U)\}$ is closed under unions and intersections, that is, it is a* lattice family*.*

Proof. Exercise. □

Consider the optimization problem over polymatroids:

- Given: $f : 2^S \to \mathbb{R}$ submodular, $w : S \to \mathbb{R}$, and say $n = |S|$.
- Goal: $\max_{x \in EP_f} \sum_{e \in S} w_e x_e$

We assume that $EP_f \neq \varnothing$, that is, $f(\varnothing) \geqslant 0$. In fact, we can assume $f(\varnothing) = 0$, otherwise we can define $\bar{f}(U) = \begin{cases} f(U) & \forall U \neq \varnothing \\ 0 & \text{if } U = \varnothing \end{cases}$, then $\bar{f}$ is submodular with $EP_{\bar{f}} = EP_f$.

If there are negative weights, then $x = -\infty$ is feasible and achieves the optimum. So we can assume $w \geqslant 0$.

Algorithm:

1. Order the elements $s_1, \ldots, s_n$ such that $w(s_i) \geqslant w(s_{i+1})$. Let $U_i = \{s_1, \ldots, s_i\}$, $U_0 = \varnothing$.
2. Let $X \in \mathbb{R}^S$, $X_{s_i} := f(U_i) - f(U_{i-1})$ $\forall i = 1, \ldots, n-1$,

   $Y \in \mathbb{R}^{2^S}$, $Y(U_i) := \begin{cases} w(s_i) - w(s_{i+1}) & \forall i = 1, \ldots, n \\ w(s_n) & \text{if } i = n \end{cases}$, $Y(T) = 0$ otherwise.

Theorem 2.7 (Edmonds)**.** *Let $f : 2^S \to \mathbb{R}$ be submodular, $f(\varnothing) = 0$, $w : S \to \mathbb{R}_+$. Then X and Y above are optimal primal and dual solutions to*

$$\max \sum_{e \in S} w_e x_e \qquad\qquad \min \sum_{T \subseteq S} y(T) f(T)$$

$$x \in EP_f \qquad\qquad \sum_{T \subseteq S, s_i \in T} y(T) = w(s_i) \ \forall i = 1, \ldots, n$$

$$y \geqslant 0$$

Proof. First, we show that X is primal feasible. To show $X(U) \leqslant f(U)$ $\forall U \subseteq S$: by induction on $|U|$. The base case is trivial. Let k be the largest index such that $s_k \in U$. By induction hypothesis, $x(U - s_k) \leqslant f(U - s_k)$. Then $x(U) = x(U - s_k) + x(s_k) \leqslant f(U - s_k) + f(U_k) - f(U_{k-1})$.

We want to show $x(e) \leqslant f(U)$. Observe that $U - s_k = U \cap U_{k-1}$ and $U_k = U \cup U_{k-1}$, so we can write $f(U \cap U_{k-1}) + f(U \cup U_{k-1}) \leqslant f(U) + f(U_{k-1})$, as desired.

Next, Y is dual feasible: the nonnegativity of Y is trivial, and $\sum_{T \subseteq S : s_i \in T} = \sum_{j=i}^{n} Y(U_j) = w(s_i)$.

Consider the objective value of X:

$$\sum_{i=1}^{n} w(s_i) X(s_i) = \sum_{i=1}^{n} w(s_i)(f(U_i) - f(U_{i-1})) = \sum_{i=1}^{n-1} f(U_i)(w(s_i) - w(s_{i+1})) + f(U_n) w(s_n) = \sum_{U \subseteq S} f(U) Y(U)$$

which shows optimality. □

Observation 2.8. *If f is integral, then X is integral.*
Theorem 2.9. *The system defining EP_f is TDI.*
Definition 2.10. $f : 2^S \to \mathbb{R}$ is *non-decreasing*[1] if $\forall U \subseteq U' \subseteq S$, $f(U) \leqslant f(U')$.
Observation 2.11. *If $f : 2^S \to \mathbb{R}$ is non-decreasing submodular with $f(\varnothing) = 0$, $w : S \to \mathbb{R}_+$, then X obtained from the algorithm is optimal to* $\max\left\{w^T x : x \in P_f\right\}$.
Corollary 2.12. *If $f : 2^S \to \mathbb{R}$ is non-decreasing submodular, then the system defining P_f is TDI.*

[1] Often *monotone* in the literature.

Lecture 23, November 4, 2015

1 Submodular Functions, continued

Recall the two polyhedra associated with a submodular function.
Definition 1.1. The *polymatroid* associated with a submodular function $f : 2^S \to \mathbb{R}$ is

$$P_f = \left\{ x \in \mathbb{R}^S : x_e \geqslant 0 \ \forall e \in S, x(U) \leqslant f(U) \ \forall U \subseteq S \right\}.$$

The *extended polymatroid* associated with f is

$$EP_f = \left\{ x \in \mathbb{R}^S : x(U) \leqslant f(U) \ \forall U \subseteq S \right\}.$$

Also recall:
Theorem 1.2 (Edmonds)**.** *A greedy algorithm solves the problem* $\max \left\{ \sum_{e \in S} w_e x_e : x \in EP_f \right\}$.
Theorem 1.3. *If $f : 2^S \to \mathbb{R}$ is submodular, $f(\varnothing) = 0$, then for all $U \subseteq S$, $f(U) = \alpha(U)$ where $\alpha(U) = \max \left\{ \sum_{e \in U} x_e : x \in EP_f \right\}$.*

Proof. $\alpha(U) \leqslant f(U)$: $\alpha(U) = X^*(U) \leqslant f(U)$.

$\alpha(U) \geqslant f(U)$: Set $w_e = \begin{cases} 1 & \forall e \in U \\ 0 & \text{otherwise} \end{cases}$. Say $U = \{s_1, \ldots, s_k\}$, $X^*(s_i) = f(U_i) - f(U_{i-1})$. Then

$$\alpha(U) = \sum_{e \in S} w_e X^*(e) = \sum_{i=1}^{k} X^*(s_i) = \sum_{i=1}^{k} f(U_i) - f(U_{i-1}) = f(U). \qquad \square$$

1.1 Application: Submodular Function Minimization

- Given: S, say $|S| = n$, $f : 2^S \to \mathbb{R}$.
- Goal: $\min_{U \subseteq S} f(U)$.

Without loss of generality we may assume $f(\varnothing) = 0$.
Observation 1.4. *The greedy algorithm solves optimization over EP_f efficiently, so by the ellipsoid algorithm, we can solve the separation problem over EP_f efficiently. Then we can also solve the separation problem over $Q_f = EP_f \cap \left\{ x \in \mathbb{R}^S : x_e \leqslant 0 \ \forall e \in S \right\}$, and so again by the ellipsoid algorithm, we can solve optimization over Q_f efficiently.*
Lemma 1.5. *If $f(\varnothing) = 0$, then $\min_{U \subseteq S} f(U) = \max \left\{ \sum_{e \in S} x_e : x \in Q_f \right\}$.*

Proof. Define $g : 2^S \to R$, $g(U) := \min_{T \subseteq U} f(T)$.

Claim 1.6. *g is submodular.*

Proof. Exercise. ■

Claim 1.7. $EP_g = Q_g$.

Proof. Let $x \in EP_g$. Let $e \in S$, then $x_e \leqslant g(\{e\}) = \min\{f(\varnothing), f(\{e\})\} \leqslant f(\varnothing) = 0$. Let $U \subseteq S$, then $x(U) \leqslant g(U) = \min_{T \subseteq U} f(T) \leqslant f(U)$. So $x \in Q_g$.
Let $x \in Q_g$. We need to show $x(U) \leqslant g(U)$ $\forall U \subseteq S$. Let $U \subseteq S$. Let $T = \arg\min_{T \subseteq U} f(T)$, then $x(U) = x(T) + x(U - T) \leqslant f(T) = g(U)$. So $x \in EP_g$. ■

Then by Theorem 1.3, $g(\varnothing) = 0 \implies g(S) = \max\left\{\sum_{e \in S} : x \in EP_g\right\} = \max\left\{\sum_{e \in S} x_e : x \in Q_f\right\}$. □

This gives an algorithm to find a minimizer:

1. Find $\alpha = \min_{U \subseteq S} f(U)$
2. Repeat:
   (a) Find $s \in S$ such that $\min_{U \subseteq S - s} f(U) = \alpha$.
   (b) $S \leftarrow S - s$
   (c) If there is no such $s \in S$, then output S.

2 Arborescences

Recall:
Definition 2.1. A *branching* is a directed forest where each vertex in-degree $\leqslant 1$.
Definition 2.2. An *arborescence* is a branching with exactly one vertex with in-degree 0. The vertex r with in-degree 0 is the *root*, and the arborescence is called an r-arborescence.

So an arborescence generalizes a spanning tree.
Proposition 2.3. *Let $G = (V, E)$ be directed, $v \in V$. G contains an r-arborescence iff for each $v \in V - r$, there exists an $r \rightsquigarrow v$ directed path in G.*

Proof. ($\implies$) is easy. Depth-First Search gives ($\impliedby$). □

Definition 2.4. A subgraph H of a directed graph is *strongly connected* if for each $u, v \in V(H)$, there exists a $u \rightsquigarrow v$ path and a $v \rightsquigarrow u$ path in H.

Strongly connected components of a graph can be found in $O(m)$ time via Depth-First Search.
Corollary 2.5. *Let G be a directed graph, $C_1, \ldots, C_k$ be the strongly connected components. Then G contains an r-arborescence iff there exists at most one component C_i such that $\delta^{in}(C_i) = \varnothing$ and if such a C_i exists, then $r \in C_i$.*

Problem:

- Given $G = (V, E)$, $w : E \to \mathbb{R}_+$, $r \in V$
- Goal: Find a minimum w-weight r-arborescence.

Algorithm:

1. $A_0 = \{e \in E : w(e) = 0\}$.
2. If (V, A_0) contains an r-arborescence T, return T.
3. Else there exists a strongly connected component C of (V, A_0) such that $r \notin C$ and $\forall e \in \delta^{\text{in}}_G(C)$, $w(e) > 0$.
4. Let $\alpha := \min_{e \in \delta^{\text{in}}_G(C)} w(e)$, $w'(e) := \begin{cases} w(e) - \alpha & \forall e \in \delta^{\text{in}}_G(C) \\ w(e) & \text{otherwise} \end{cases}$. Then $|A'_0| > |A_0|$.

5. Recurse on G for w' and obtain T' as a minimum w'-weight r-arborescence.
6. Refine T' to get T'' with $w'(T'') \leqslant w'(T')$ and T'' has exactly one edge of $\delta_G^{\text{in}}(C)$.
7. Return T''.

Theorem 2.6 (Chu-Liu)**.** *This algorithm finds an optimal r-arborescence.*

Proof. Let B be an r-arborescence. Then

$$w(B) \geqslant w'(B) + \alpha|B \cap \delta_G^{\text{in}}(C)| \geqslant w'(B) + \alpha \geqslant w'(T') + \alpha \geqslant w'(T'') + \alpha = w(T'') \qquad \square$$

Running-time: Step (1)-(4) and (6) can be done $O(|E|)$ time each. The depth of the recursion in Step (5) is $O(|E|)$. So the overall runtime is $O(|E|^2)$.

Definition 2.7. Let $G = (V, E)$ be directed, $r \in V$. An *r-cut* is a subset $X \subseteq E$ for which there exists $U \subseteq V$ such that $r \notin U$ and $\delta^{\text{in}}(U) \subseteq X$.

Observation 2.8.

- *Every r-cut intersects every r-arborescence.*
- *If X intersects every r-arborescence, then X contains an r-cut.*
- *Inclusionwise minimal sets of arcs intersection every r-arborescence are exactly the minimal r-cuts.*

This suggests a min-max relation.

Theorem 2.9. *Given $G = (V, E)$ directed, $r \in V$, $c : E \to \mathbb{Z}_+$, then the minimum c-weight of an r-arborescence is equal to the size of a maximum collection of r-cuts such that each edge e appears in $c(e)$ of these cuts.*

Lecture 24, November 5, 2015

1 Arborescences, continued

Recall:
Definition 1.1. An *r-arborescence* is a directed forest where each vertex in-degree $\leqslant 1$ and r has in-degree 0.
Definition 1.2. Let $G = (V, E)$ be directed, $r \in V$. An *r-cut* is a subset $X \subseteq E$ for which there exists $U \subseteq V$ such that $r \notin U$ and $\delta^{\text{in}}(U) \subseteq X$.
Observation 1.3. *Every r-cut intersects every r-arborescence.*

This suggests a min-max relation.
Theorem 1.4. *Given $G = (V, E)$ directed, $r \in V$, $c : E \to \mathbb{Z}_+$, then the minimum c-cost of an r-arborescence is equal to the size of a maximum collection of r-cuts such that each edge e appears in $c(e)$ of these cuts.*

Proof. ($\geqslant$): Let $R_1, R_2, \ldots, R_k$ be r-cuts such that each edge is in at most $c(e)$ of these cuts.

Look at the simpler case where $c(e) = 1$ $\forall e \in E$, $R_1, \ldots, R_k$ are all disjoint. Then by the observation, the minimum c-cost of an r-arborescence is $\geqslant k$.

In the general case, let T be a minimum c-cost r-arborescence. Then

$$c(T) = \sum_{e \in E} c(e) \geqslant \sum_{e \in T} \sum_{i=1}^{k} \mathbb{1}_{e \in R_i} = \sum_{i=1}^{k} \sum_{e \in T} \mathbb{1}_{e \in R_i} \geqslant \sum_{i=1}^{k} 1 = k.$$

($\leqslant$): By induction on $\sum_{e \in E} c(e)$. Set $A_0 = \{e \in E : c(e) = 0\}$. The interesting case is when there does not exist an r-arborescence in (V, A_0). Then there is a strongly connected component K such that $r \notin V(K)$ and $\forall e \in \delta^{\text{in}}(K)$, $c(e) \geqslant 1$. Define

$$c'(e) := \begin{cases} c(e) - 1 & \forall e \in \delta^{\text{in}}(K) \\ c(e) & \text{otherwise} \end{cases}$$

so $\sum_{e \in E} c'(e) < \sum_{e \in E} c(e)$.

Let B be a minimum c'-cost r-arborescence. By the algorithm, $|B \cap \delta^{\text{in}}(K)| = 1$. By induction, there exist r-cuts $R_1, \ldots, R_p$ such that $p = c'(B)$, and each edge e appears in at most $c'(e)$ of these cuts. Note that $c(B) = c'(B) + 1$, and so there exists r-cuts $R_1, \ldots, R_p, \delta_G^{\text{in}}(K)$ such that each edge appears in at most $c(e)$ of these cuts. □

1.1 Polyhedral aspects

Definition 1.5. $P_{r\text{-arb}}(G) :=$ convex hull of incidence vectors of r-arborescences in G.
Theorem 1.6. $P_{r\text{-}arb}(G) + \mathbb{R}^E_{\geqslant 0} = \{x \in \mathbb{R}^E : x(C) \geqslant 1 \ \forall r\text{-}cuts\ C, x_e \geqslant 0 \ \forall e \in E\}$.
Definition 1.7. The *up-hull* of the r-arborescence polytope is $P^{\uparrow}_{r\text{-arb}}(G) := P_{r\text{-arb}}(G) + \mathbb{R}^E_{\geqslant 0}$.
Corollary 1.8. $P_{r\text{-}arb}(G) = \{x \in \mathbb{R}^E : x(C) \geqslant 1 \ \forall r\text{-}cuts\ C, x_e \geqslant 0 \ \forall e \in E, x(\delta^{in}(v)) = 1 \ \forall v \in V - r, x(\delta^{in}(r)) = 0\}$.

Theorem 1.9. *Let $r_1, \ldots, r_k \in V(G)$. There exist k arc-disjoint arborescences $B_1, \ldots, B_k$ where* $\operatorname{root}(B_i) = r_i$ *iff* $|\delta^{in}(U)| \geqslant |\{i : r_i \notin U\}|\ \forall U \subseteq V$.

Let us prove a more general version of the theorem.

Definition 1.10. A *branching* is a directed forest with each component having exactly one vertex with in-degree 0 and every other vertex has in-degree 1. Call the vertices with in-degree 0 $\operatorname{roots}(B)$. A branching B is *spanning* if $V(B) = V$.

Theorem 1.11. *Let $R_1, \ldots, R_k \subseteq V$. There exist k arc-disjoint spanning branchings B_i with* $\operatorname{roots}(B_i) = R_i$ *iff* $|\delta^{in}(U)| \geqslant |\{i : R_i \cap U = \varnothing\}|\ \forall U \subseteq V$.

Proof. We have already shown ($\implies$).

($\impliedby$): By induction on $\sum_{i=1}^{k} |V - R_i|$.

Base Case: Say $\sum_{i=1}^{k} |V - R_i| = 0$, then $V = R_1 = \cdots = R_k$, so $B_1 = \cdots = B_k = \varnothing$, which are arc-disjoint.

Now say $\sum_{i=1}^{k} |V - R_i| \geqslant 1$, so without loss of generality say $R_1 \neq V$. Define $g(U) := |\{i : R_i \cap U = \varnothing\}|$ for all $U \subseteq V$.

Pick an inclusionwise minimal $W \subseteq V$ such that (1) $W \cap R_1 \neq \varnothing$, (2) $W - R_1 \neq \varnothing$, (3) $|\delta^{\text{in}}(W)| = g(W)$. Such a set exists since $W = V$ is a choice. Then

$$\left|\delta^{\text{in}}_G(W - R_1)\right| \geqslant g(W - R_1) > g(W) = \left|\delta^{\text{in}}(W)\right|$$

which means that there is an arc $\vec{uv}$ where $u \in R_1$, $v \in W - R_1$. Set $R_1' = R_1 \cup \{v\}$.

Claim 1.12. $|\delta^{in}_{G-\vec{uv}}(U)| \geqslant g'(U)$ *where $g'(U)$ is the number of sets in $R_1', R_2, \ldots, R_k$ disjoint from U.*

Proof. Suppose not, then there exists some $U \subseteq V$ such that $|\delta^{\text{in}}_{G-\vec{uv}}(U)| < g'(U)$, then

$$\left|\delta^{\text{in}}_{G-\vec{uv}}(U)\right| < g'(U) \leqslant g(U) \leqslant \left|\delta^{\text{in}}_G(U)\right|.$$

Since $\left|\delta^{\text{in}}_{G-\vec{uv}}(U)\right| \in \left\{\left|\delta^{\text{in}}_G(U)\right|, \left|\delta^{\text{in}}_G(U)\right| - 1\right\}$, we obtain $\left|\delta^{\text{in}}_{G-\vec{uv}}(U)\right| = \left|\delta^{\text{in}}_G(U)\right| - 1$. This means that $|uv| \in \delta^{\text{in}}_G(U)$, in particular, $U \cap R_1' \neq \varnothing$. Furthermore, we have $g'(U) = g(U)$, so $U \cap R_1 \neq \varnothing$, and $g(U) = \left|\delta^{\text{in}}_G(U)\right|$.

We will show that $U \cap W$ satisfies (1), (2), and (3) but $|U \cap W| < |W|$, contradicting the choice of W. Since $u \in W, u \notin U$, $U \cap W \subsetneq W$. To show (3):

$$\begin{aligned} g(U \cap W) \leqslant \left|\delta^{\text{in}}_G(U \cap W)\right| &\leqslant \left|\delta^{\text{in}}_G(U)\right| + \left|\delta^{\text{in}}_G(W)\right| - \left|\delta^{\text{in}}_G(U \cup W)\right| \\ &\leqslant g(U) + g(W) - g(U \cup W) \leqslant g(U \cap W) \end{aligned}$$

in particular, $g(U \cap W) = \left|\delta^{\text{in}}_G(U \cap W)\right|$.

For (1), $U \cap R_1 \neq \varnothing$, and $W \cap R_1 \neq \varnothing$, so $(U \cup W) \cap R_1 \neq \varnothing$. Since $g(U) + g(W) - g(U \cup W) = g(U \cap W)$, we conclude $(U \cap W) \cap R_1 \neq \varnothing$.

Finally for (2), $v \in (U \cap W) - R_1$ so $(U \cap W) - R_1 \neq \varnothing$. ■

Then since $\sum_{i=2}^{k} |V - R_i| + |V - R_1'| < \sum_{i=1}^{k} |V - R_i|$, by the induction hypothesis, there exist arc-disjoint spanning branchings $B_1', B_2, \ldots, B_k$ in $G - \vec{uv}$ such that $\operatorname{roots}(B_i) = R_i$ for $i = 2, \ldots, k$ and $\operatorname{roots}(B_1') = R_1'$. Set $B_1 = B_1' \cup \vec{uv}$. □

Corollary 1.13. *The minimum size of an r-cut is equal to the maximum number of arc-disjoint r-arborescences.*

Lecture 25, November 10, 2015

1 Connectivity in Graphs and Submodular Flows

We wish to understand which graphs have good connectivity properties and how to improve connectivity in graphs.

Recall:
Definition 1.1. G is *strongly edge-connected* if $\forall u, v \in V$, there exists a $u \rightsquigarrow v$ path and a $v \rightsquigarrow u$ path in G. Equivalently, $\forall \varnothing \neq U \subseteq V$, $|\delta^{\text{in}}(U)| \geqslant 1$ and $|\delta^{\text{out}}(U)| \geqslant 1$.

Note that the underlying undirected (multi-)graph is 2-edge-connected, $|\delta_{\text{undir}(G)}| \geqslant 2$. In fact, a sort-of converse holds: if an undirected graph is 2-edge-connected then there exists an orientation that is strongly connected. This is because the graph has an ear decomposition, then we can orient all of the edges in the cycle the same direction, and then for each ear, orient all edges of the ear in the same direction. This is due to Robbins.

A more general theorem holds.
Definition 1.2. $G = (V, E)$ is *strongly k-edge connected* if $\forall u, v \in V$, there exist k disjoint $u \rightsquigarrow v$ paths, and k disjoint $v \rightsquigarrow u$ paths. Equivalently, $\forall \varnothing \neq U \subseteq V$, $|\delta^{\text{in}}(U)| \geqslant k$ and $|\delta^{\text{out}}(U)| \geqslant k$.
Theorem 1.3 (Nash-Williams)**.** *Let G be an undirected graph. G is $2k$-edge-connected iff there exists an orientation of the edges of G that gives a strongly k-edge connected graph.*

Before we show this, let us develop a general framework for dealing with connectivity in graphs. Proving this result through the framework becomes very easy.
Definition 1.4. Given $G = (V, E)$ directed:

1. A family $\mathcal{F} \subseteq 2^V$ is *cross-free* if $\forall A, B \in \mathcal{F}$, $A \subseteq B$ or $B \subseteq S$ or $A \cap B = \varnothing$ or $A \cup B = V$.
   Example 1.5. Laminar families are cross-free.

2. A family $\mathcal{C} \subseteq 2^V$ is *crossing* if $\forall A, B \in \mathcal{C}$, if $A \cap B \neq \varnothing$ and $A \cup B \neq V$, then $A \cap B, A \cup B \in C$.
   Example 1.6. Fix $s, t \in V$, $\mathcal{C} = \{U \subseteq V : s \in U, t \notin U\}$.

3. A function $f : \mathcal{C} \to \mathbb{R}$ is *submodular on crossing pairs* or *crossing submodular* if $\forall T, U \in \mathcal{C}$ with $T \cap U \neq \varnothing$ and $T \cup U \neq V$, we have $f(T) + f(U) \geqslant f(U \cap T) + f(U \cup T)$.

4. Given graph G, crossing family $\mathcal{C}$, crossing submodular f, a *submodular flow* is $x \in \mathbb{R}^E$ such that the flow that sinks in a set U is bounded f, i.e., $x(\delta^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \leqslant f(U)\ \forall U \in \mathcal{C}$, $U \neq \varnothing, V$.

5. The *submodular flow polyhedron* is given by $\left\{x \in \mathbb{R}^E : x(\delta^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \leqslant f(U)\ \forall U \in \mathcal{C}, U \neq \varnothing, V\right\}$.

Theorem 1.7 (Edmonds-Giles)**.** *The system of submodular flow constraints is box-TDI, i.e., $\forall \ell, u : E \to \mathbb{Z}$, $\forall w : E \to \mathbb{Z}$, the dual to the LP*

$$\begin{aligned} \max \sum_{e \in E} w_e x_e & \\ x(\delta^{in}(U)) - x(\delta^{out}(U)) \leqslant f(U) & \qquad \forall U \in \mathcal{C}, U \neq \varnothing, V \\ \ell(e) \leqslant x_e \leqslant w(e) & \qquad \forall e \in E \end{aligned}$$

has an integral optimum.

Proof. Fix $\ell, u, w : E \to \mathbb{Z}$. The dual is

$$\begin{aligned} \min \; & \sum_{e \in E} (z_1(e)u(e) - z_2(e)\ell(e)) + \sum_{\substack{U \in \mathcal{C} \\ U \neq \varnothing, V}} y(U) f(U) \\ & \sum_{\substack{U \in \mathcal{C} \\ U \neq \varnothing, V \\ e \in \delta^{\text{in}}(U)}} y(U) - \sum_{\substack{U \in \mathcal{C} \\ U \neq \varnothing, V \\ e \in \delta^{\text{out}}(U)}} y(U) \geqslant w_e \qquad \forall e \in E \\ & z_1, z_2, y \geqslant 0 \end{aligned}$$

The approach will be similar to matroid intersection: show that a subsystem is TU. Define (z_1, z_2, y) dual optimum that minimizes the potential function $\Phi(y) = \sum_{U \in \mathcal{C}} y(U)|U||V - U|$. We will look at the support of y.

Let $\mathcal{C}_0 = \{U \in \mathcal{C} : y(U) > 0\}$.

Claim 1.8. *$\mathcal{C}_0$ is cross-free.*

Proof. Suppose not. Then there exist $A, B \in \mathcal{C}_0$ such that $A \not\subseteq B$, $B \not\subseteq A$, $A \cap B \neq \varnothing$, $A \cup B \neq V$, and $y(A), y(B) > 0$. Take $\alpha = \min\{y(A), y(B)\}$. Set

$$y'(T) = \begin{cases} y(T) - \alpha & \text{if } T = A \text{ or } T = B \\ y(T) + \alpha & \text{if } T = A \cap B \text{ or } T = A \cup B \\ y(T) & \text{otherwise} \end{cases}$$

1. y' is dual feasible.
2. $\text{ObjVal}(z_1, z_2, y') - \text{ObjVal}(z_1, z_2, y) = \alpha(f(A \cup B) + f(A \cap B) - f(A) - f(B)) \leqslant 0$.
3. $\Phi(y') - \Phi(y) < 0$.

This contradicts the choice of y. ■

Claim 1.9. *The constraint matrix corresponding to $\mathcal{C}_0$ is TU.*

This immediately gives an integral optimum (z_1, z_2, y).

Proof Sketch. Let $\mathcal{C}_0 = \{U_1, \ldots, U_k\}$. The transpose of the constraint matrix is M where the rows correspond to sets in $\mathcal{C}_0$ and columns correspond to edges in E. So

$$M[U, e] = \begin{cases} +1 & \text{if } e \text{ is incoming into } U \\ -1 & \text{if } e \text{ is outgoing from } U \\ 0 & \text{otherwise} \end{cases}$$

As a simple case, take $\mathcal{C}_0$ to be laminar. A laminar family induces a tree $T = (\mathcal{C}_0 \cup \{V\}, A)$ where $A = \left\{ \overrightarrow{U_i U_j} : \; U_j \text{ is an inclusionwise maximal set of } \mathcal{C}_0 \text{ contained in } U_i \right\}$.

Let $\pi(v)$ be the smallest set in $\{V\} \cup \mathcal{C}_0$ containing v.

1. $\left\{U \in \mathcal{C}_0 \cup \{V\} : \vec{xy} \in \delta^{\text{in}}(U)\right\} = \{U \in \mathcal{C}_0 \cup \{V\} :$ the unique incoming edge into U in T is traversed in the forward direction by the unique path $\pi(x) \rightsquigarrow \pi(y)$ in $T\}$.
2. $\{U \in \mathcal{C}_0 \cup \{V\} : \vec{xy} \in \delta^{\text{out}}(U)\} = \{U \in \mathcal{C}_0 \cup \{V\}$: the unique incoming edge into U in T is traversed in the backward direction by the unique path $\pi(x) \rightsquigarrow \pi(y)$ in $T\}$.

Matrices of this form are *network matrices*, which are TU (see below). ■

□

2 General Construction of Network Matrices

Let $T = (S, A)$ be a directed tree, spanning, and $H = (S, F)$ a directed graph.

A *network matrix* for T, H is a matrix $N \in \{0, \pm 1\}^{|A| \times |F|}$, where rows correspond to edges of T and columns correspond to edges of H.

For $\vec{xy} \in F$, let P be the unique path in T from x to y.

$$N(\vec{ab}, \vec{xy}) = \begin{cases} +1 & \text{if } \vec{ab} \in E(P) \text{ and is traversed by } P \text{ in the direction of } T \\ -1 & \text{if } \vec{ab} \in E(P) \text{ and is traversed by } P \text{ in the opposite direction of } T \\ 0 & \text{otherwise} \end{cases}$$

So in the Proof Sketch of Claim 1.9, if $H = (\mathcal{C}_0 \cup \{V\}, F)$ where $F = \left\{ \vec{U_i U_j} : \exists \vec{uv}, U_i = \pi(u), U_j = \pi(v) \right\}$ then M is the network matrix of T, H.

Theorem 2.1 (Tutte)**.** *Network Matrices are TU.*

Lecture 26, November 12, 2015

1 Applications of the Edmonds-Giles Theorem

Recall:

Given directed graph G, crossing family $\mathcal{C}$, crossing submodular f:
Theorem 1.1 (Edmonds-Giles)**.** *For all $\ell, u : E \to \mathbb{Z}$, $\forall w : E \to \mathbb{Z}$, the system*

$$\begin{aligned} x(\delta^{in}(U)) - x(\delta^{out}(U)) &\leqslant f(U) && \forall U \in \mathcal{C}, U \neq \varnothing, V \\ \ell(e) \leqslant x_e &\leqslant w(e) && \forall e \in E \end{aligned}$$

is TDI.

This is the mother of many classic results.

1.1 Connectivity in Graphs

Theorem 1.2 (Nash-Williams)**.** *Let G be an undirected graph. G is $2k$-edge-connected iff there exists an orientation of the edges of G that gives a strongly k-edge connected graph.*

Proof. ($\Longleftarrow$) is easy.

($\Longrightarrow$): Pick an arbitrary orientation of the edges of G to get $\vec{G}$. If $\vec{G}$ is strongly k-edge connected then we are done. Suppose not. We would like to know which edges to reverse. We will set this up as an Integer Program in such a fashion that it will tell us which edges to reverse. We will use the Edmonds-Giles theorem, so we need to define a crossing family $\mathcal{C}$ and crossing submodular f.

Let $\mathcal{C} = \{U \subseteq V : U \neq \varnothing, V\}$, $f(U) = |\delta^{\text{in}}_{\vec{G}(U)}| - k$. By Theorem 1.1,

$$\begin{aligned} x(\delta^{\text{in}}_{\vec{G}}) - x(\delta^{\text{out}}_{\vec{G}}(U)) &\leqslant |\delta^{\text{in}}_{\vec{G}}(U)| - k && \forall U \in \mathcal{C}, U \neq \varnothing, V \\ 0 \leqslant x_e &\leqslant 1 && \forall e \in E \end{aligned}$$

is TDI. So the polytope $P = \big\{x \in \mathbb{R}^E : x \text{ satisfies the two constraints}\big\}$ is integral.

Claim 1.3. *$P \neq \varnothing$.*

Proof. Set $x_e = \frac{1}{2}$ $\forall e \in E$. Then since $|\delta^{\text{in}}_{\vec{G}}(U)| - |\delta^{\text{out}}_{\vec{G}}(U)| = |\delta_G(U)| \geqslant 2k$,

$$x(\delta^{\text{in}}_{\vec{G}}) - x(\delta^{\text{out}}_{\vec{G}}(U)) = \frac{1}{2}\left(|\delta^{\text{in}}_{\vec{G}}(U)| - |\delta^{\text{out}}_{\vec{G}}(U)|\right) \leqslant |\delta^{\text{in}}_{\vec{G}}(U)| - k.$$ ■

Let $\bar{x}$ be an integral vertex of P. Obtain $\vec{H}$ from $\vec{G}$ by reversing the arcs e such that $x_e = 1$.

Fix $U \subseteq V$, $U \neq \varnothing, V$. Then

$$|\delta^{\text{in}}_{\vec{H}}(U)| - |\delta^{\text{in}}_{\vec{G}}(U)| = \bar{x}(\delta^{\text{out}}_{\vec{G}}(U)) - \bar{x}(\delta^{\text{in}}_{\vec{G}}(U)) \geqslant k - |\delta^{\text{in}}_{\vec{G}}(U)|$$

so $|\delta^{\text{in}}_{\vec{H}}(U)| \geqslant k$. Also, $|\delta^{\text{out}}_{\vec{H}}(U)| = |\delta^{\text{in}}_{\vec{H}}(V - U)| \geqslant k$. So $\vec{H}$ is strongly k-edge-connected. □

1.2 Directed Cuts

Definition 1.4. Given a directed graph $D = (V, E)$, a subset C of edges a *directed cut* if there exists $U \subseteq V$ such that $\delta^{\text{out}}(U) = \varnothing$, and $\delta^{\text{in}}(U) = C$.
Observation 1.5. *There exists a directed cut in D iff D is not strongly connected.*

We can verify if a given subset of edges $F \subseteq E$ is a directed cut efficiently using a reachability test such as DFS.

In applications such as network reliability, we want to bound the number of arc-disjoint directed cuts.
Definition 1.6. A *directed cut cover* is a subset of edges that intersects every directed cut.

From the definition, it is clear that the maximum number of arc-disjoint directed cuts is upper bounded by the minimum size of a directed cut cover. It turns out that this min-max relation is tight. Before we prove this, let us consider how to verify if a given set of edges forms a directed cut cover.
Definition 1.7. A directed graph is *weakly connected* if the underlying undirected graph is connected.
Proposition 1.8. *Suppose $D = (V, E)$ is a weakly connected directed graph, and $B \subseteq E$. Then B is a directed cut cover iff $D' = (V, E \cup B')$, where $B' = \{\vec{vu} : \vec{uv} \in B\}$, is strongly connected.*

Proof.

($\implies$): Suppose D' is not strongly connected. Then there exists $U \subseteq V$ with $\delta_{D}^{\text{out}}(U) = \varnothing$. This exists in D as well, so $B \cap \delta_{D}^{\text{in}}(U) \neq \varnothing$, so there exists an arc out of U in D', a contradiction.

($\impliedby$): Suppose B is not a directed cut cover. Then there exists a directed cut $U \subseteq V$ such that $B \cap \delta^{\text{in}}(U) = \varnothing$. Then this directed cut should also exist in D', which means that D' is not strongly connected. Contradiction. □

Theorem 1.9 (Lucchesi-Younger)**.** *Let $D = (V, E)$ be weakly connected. The maximum number of arc-disjoint directed cuts is equal to the minimum size of a directed cut cover.*

Proof. Set $\mathcal{C} = \{U \subseteq V : \delta^{\text{out}}(U) = \varnothing, U \neq \varnothing, V\}$.

Claim 1.10. *$\mathcal{C}$ is a crossing family.*

Proof. Trivial. ■

Set $f : \mathcal{C} \to \mathbb{R}$ to be $f(U) = -1$. Take $u(e) = 0$, with no lower bound. Then Theorem 1.1 implies that the system

$$\begin{aligned} x(\delta_{\vec{G}}^{\text{in}}) &\leqslant -1 && \forall U \in \mathcal{C}, U \neq \varnothing, V \\ x_e &\leqslant 0 && \forall e \in E \end{aligned}$$

is TDI. So the LP

$$\begin{aligned} \min\ -\sum_{e \in E} x_e & \\ x(\delta_{\vec{G}}^{\text{in}}) &\leqslant -1 && \forall U \in \mathcal{C}, U \neq \varnothing, V \\ x_e &\leqslant 0 && \forall e \in E \end{aligned}$$

or, if we multiply all of the variables by -1,

$$\begin{aligned} \min\ \sum_{e \in E} x_e & \\ x(\delta_{\vec{G}}^{\text{in}}) &\geqslant 1 && \forall U \in \mathcal{C}, U \neq \varnothing, V \\ x_e &\geqslant 0 && \forall e \in E \end{aligned}$$

has integral optimum. We can assume that the optimum is a 0-1 value. The objective value of the optimum is the minimum size of a directed cut cover.

Let us write down the dual LP:

$$
\begin{aligned}
\max \sum_{\substack{U \in \mathcal{C} \\ U \neq \varnothing, V}} y(U) & \\
\sum_{\substack{U \in \mathcal{C} \\ U \neq \varnothing, V \\ e \in \delta^{\text{in}}(U)}} y(U) \leqslant 1 & \qquad \forall e \in E \\
y \geqslant 0 &
\end{aligned}
$$

Since the system is TDI, the dual has an integral optimum, so the objective value of the dual optimum is the maximum number of arc-disjoint directed cuts. □

The following is open:

Conjecture 1.11. *The maximum number of arc-disjoint directed cut covers is equal to the minimum size of a directed cut.*

The minimum size of a directed cut can be found efficiently. The complexity of finding the maximum number of arc-disjoint directed cut covers is open.

Lecture 27, December 1, 2015

1 Network Matrices, Revisited

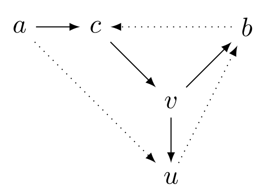
Recall:

Let $T = (V, A)$ be a directed tree that is spanning, and $H = (V, F)$ a directed graph.

A *network matrix* for T, H is a matrix $N \in \{0, \pm 1\}^{|A| \times |F|}$, where rows correspond to edges of T and columns correspond to edges of H, as follows: for $\vec{au} \in F$, let P be the unique $a \rightsquigarrow u$ path in T, then

$$N[\vec{xy}, \vec{au}] = \begin{cases} +1 & \text{if } \vec{xy} \in P \text{ and it is used in the forward direction} \\ -1 & \text{if } \vec{xy} \in P \text{ and it is used in the backward direction} \\ 0 & \text{otherwise} \end{cases}$$

Example 1.1. In the following, solid lines are in T and dotted lines are in H.



$$N = \begin{array}{c} \\ \vec{ac} \\ \vec{cv} \\ \vec{vu} \\ \vec{vb} \end{array} \begin{array}{c} \begin{array}{ccc} \vec{au} & \vec{ub} & \vec{bc} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$$

Recall the theorem:
Theorem 1.2 (Tutte)**.** *Network matrices are TU.*

We will now prove the theorem.

Proof. Observe that the following operations preserve the network matrix property:

1. Multiplying columns/rows by -1 (corresponds to switching directions of edges)
2. Deleting columns/rows (corresponds to deleting edges in H and contracting, respectively), in particular, submatrices of network matrices are also network matrices.

We will show that a square $n \times n$ network matrix M has determinant 0 or ± 1. We will use induction on n.

Let u be a leaf of T with neighbor v, say $\vec{vu} \in A$. If row$(\vec{uv})$ is all 0's or there is only one nonzero entry, then we are done. So without loss of generality, assume that row$(\vec{uv})$ has two non-zeroes, say in col(e_1) and col(e_2), with $M[\vec{vu}, e_1] = +1$, $M[\vec{vu}, e_2] = -1$ (we can achieve this by multiplying by -1 if necessary). Since u is a leaf, e_1 is an incoming edge of u, and e_2 is an outgoing edge of u. In Example 1.1, $e_1 = \vec{au}$ and $e_2 = \vec{ub}$.

Replace col$(e_2) \leftarrow$ col(e_2) + col(e_1) to obtain M'. This operation preserves determinant. In Example 1.1, this corresponds to dropping $\vec{ub}$ from H and adding $\vec{ab}$. More generally:

Claim 1.3. *M' is a network with respect to T, and H' is obtained from H by replacing e_2 with* tail$(e_1) \to$ head(e_2)*.*

Then we can just keep performing this operation for col(e_j) such that $M[\vec{uv}, e_j]$ is nonzero, $e_j \neq e_i$, until we obtain row$(\vec{uv}) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$. The rest follows by induction. $\square$

2 Multicommodity Flows

Recall the Maximum Flow problem: Given $G = (V, E)$ (directed or undirected), $s, t \in V$, capacities $c : E \to \mathbb{R}_+$,

- An $s \to t$ flow is a function $f : E \to \mathbb{R}_+$ satisfying $f(\delta^{\text{in}}(v)) = f(\delta^{\text{out}}(v))$ $\forall v \in V - \{s, t\}$,
- The *value* of the flow is $f(\delta^{\text{out}}(s))$ $(= f(\delta^{\text{in}}(t)))$,
- f is *subject to* c if $f(e) \leqslant c(e)$ $\forall e \in E$.

Definition 2.1. Given $G = (V, E)$ (directed or undirected), terminals $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ and demands $d_1, d_2, \ldots, d_k$ and capacities $c : E \to \mathbb{R}_+$, a *multiflow* is a collection $f_1, \ldots, f_k$ of flows satisfying:

1. f_i is an $s_i \to t_i$ flow of value d_i $\forall i \in [k]$
2. $\sum_{i=1}^{k} f_i(e) \leqslant c(e)$ $\forall e \in E$ (capacity contraint)

The *k-commodity multiflow problem* is to determine if there exists a feasible multiflow.

The fractional version of this problem is solvable by writing an LP. There are, however, applications in which we are interested in integral flows or edge-disjoint paths between terminals.

In the *k-disjoint paths problem (k-DP)*:

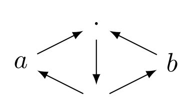
- Given $G = (V, E)$, terminal pairs $(s_1, t_1), \ldots, (s_k, t_k)$
- Does there exist vertex- (or edge-) disjoint paths $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i for $i \in [k]$?

Note that k-edge DP a special case of the integral k-multicommodity flow problem: set all the demands and capacities to 1.

We have several versions: vertex-disjoint and edge-disjoint, directed and undirected.

2.1 Reductions

1. Undirected k-vertex DP can be reduced to Directed k-vertex DP by including both $\vec{ab}$ and $\vec{ba}$ for $ab \in E$.
2. Undirected k-edge DP can be reduced to Undirected k-vertex DP as follows: for each $e \in E$, create v_e, and connect v_{e_1} and v_{e_2} if e_1 and e_2 share an endpoint; then for each terminal s_i, create a new terminal s'_i connected to each v_e where e is an edge incident to s_i. This is known as creating the Line Graph of G.
3. Directed k-edge DP can be reduced to Directed k-vertex DP in the same manner, except we only introduce an arc between v_{a_1} and v_{a_2} if head($a_{)}$ meets tail(a_2).
4. Directed k-vertex DP can be reduced to Directed k-edge DP by splitting vertices: for every vertex v create v_{in} and v_{out} with an edge $v_{\text{in}} \to v_{\text{out}}$, and redirecting all incoming edges to v to be incoming to v_{in} and all outgoing edges from v to be outgoing from v_{out}.
5. Undirected k-edge DP can be reduced to Directed k-edge DP; one way is to chain (2), (1), and (4), another is to replace an edge ab with the following gadget:



In conclusion we have the following reductions:

$$\begin{tikzcd}
\text{Undirected Edge DP} \arrow[r] \arrow[d] & \text{Undirected Vertex DP} \arrow[d] \\
\text{Directed Edge DP} \arrow[r, leftrightarrow] & \text{Directed Vertex DP}
\end{tikzcd}$$

It turns out that all of these are NP-complete (Undirected Vertex DP appears in [Karp]). However when k is fixed, the undirected versions are solvable in polynomial time; the undirected vertex version was done using graph minor theory by [Robertson-Seymour].

Lecture 28, December 3, 2015

1 Multiflows, continued

Definition 1.1. Given $G = (V, E)$ (directed or undirected), terminals $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ and demands $d_1, d_2, \ldots, d_k$ and capacities $c : E \to \mathbb{R}_+$, a *multiflow* is a collection $f_1, \ldots, f_k$ of flows satisfying:

1. f_i is an $s_i \to t_i$ flow of value d_i $\forall i \in [k]$
2. $\sum_{i=1}^{k} f_i(e) \leqslant c(e)$ $\forall e \in E$ (capacity contraint)

The *k-commodity multiflow problem* is to determine if there exists a feasible multiflow.

Note that in the undirected case, flows can go through edges in both directions, and their values are summed.

We saw that the integral version of this problem is NP-complete, and special cases correspond to various k-disjoint path problems.

1.1 Fractional Multiflow Problem

We can write an LP to solve this problem.

We introduce the variables $f_i(e)$ $\forall e \in E, i \in [k]$, and the constraints are flow conservation for each f_i at v, $\forall v \in V - \{s_i, t_i\}, i \in [k]$, demand constraints for each $i \in [k]$, and capacity constraints for each edge. Note that in the undirected case, we need to write the capacity constraints using $|f_i(e)|$, so we need to introduce new variables to linearize.

Theorem 1.2 (Onaga)**.** *There exists a feasible fractional mutiflow iff* $\forall \ell \colon E \to \mathbb{R}_+$, $\sum_{i=1}^{k} d_i \operatorname{dist}_\ell(s_i, t_i) \leqslant \sum_{e \in E} \ell(e) c(e)$*, where* $\operatorname{dist}_\ell(s_i, t_i)$ *is defined to be the shortest* s_i, t_i *distance with respect to* ℓ*. This holds in both undirected and directed cases.*

Proof. We will prove the theorem for undirected instances. The proof for directed instances is similar.

Consider all s_i, t_i paths. f_i will send some flow along the first path, some flow along the second path, and so on. We want the flow on all paths to sum up to at least d_i, and satisfy the constraints.

So let $\mathbb{P}_i$ be the set of $s_i \rightsquigarrow t_i$ paths in G. There exists a feasible multiflow iff there exists $\lambda_{i,P} \geqslant 0$ such that $\sum_{P \in \mathbb{P}_i} \lambda_{i,P} \geqslant d_i$ $\forall i \in [k]$, and $\sum_{i=1}^{k} \sum_{P \in \mathbb{P}_i} \lambda_{i,P} \chi_P(e) \leqslant c_e$ $\forall e \in E$.

Recall Farkas' Lemma: Exactly one of the following holds:

(i) $x \geqslant 0, Ax \leqslant b$,

(ii) $y \geqslant 0, y^T A \geqslant 0, y^T b < 0$.

So there exists a feasible multiflow iff $\forall y \geqslant 0$, if $y^T A \geqslant 0$, then $y^T b \geqslant 0$, where we have a row for each $i \in [k]$ and $e \in E$, and a column for each $P \in \mathbb{P}_i$.

$$A = \begin{array}{cc} & \begin{array}{ccccccc} p_1^1 & \cdots & p_{r_1}^1 & p_1^2 & \cdots & p_{r_2}^2 & \cdots \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ k \\ e_1 \\ \vdots \\ e_m \end{array} & \begin{bmatrix} -1 & \cdots & -1 & 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -1 & \cdots & -1 & 0\cdots \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix} \end{array}$$

with $A[e_j, P_k^i] = \begin{cases} 1 & \text{if } e \in P_k^i \\ 0 & \text{otherwise} \end{cases}$.

So there exists a feasible multiflow iff $\forall y = (q_1, \ldots, q_k, \ell_{e_1}, \ldots, \ell_{e_2}) \geqslant 0$, if $q_i \leqslant \sum_{e \in E : e \in P} \ell_e \; \forall P \in \mathbb{P}_i \; \forall i \in [k]$, then $\sum_{i=1}^k q_i d_i \leqslant \sum_{e \in E} \ell_e c_e$.

We can take $q_i = \min_{P \in \mathbb{P}_i} \sum_{e \in P} \ell_e = \min_{P \in \mathbb{P}_i} \ell(P) = \operatorname{dist}_\ell(s_i, t_i)$. So there exists a feasible multiflow iff for all $\ell : e \to \mathbb{R}_+$, $\sum_{i=1}^k d_i \operatorname{dist}_\ell(s_i, t_i) \leqslant \sum_{e \in E} \ell_e c_e$. $\square$

In fact, by taking a stronger form of Farkas' lemma, we can obtain a stronger form of the theorem:

Theorem 1.3 (Onaga)**.** *There exists a feasible fractional mutiflow iff* $\forall \ell \colon E \to \mathbb{Z}_+$, $\sum_{i=1}^k d_i \operatorname{dist}_\ell(s_i, t_i) \leqslant \sum_{e \in E} \ell(e) c(e)$, *where* $\operatorname{dist}_\ell(s_i, t_i)$ *is defined to be the shortest* s_i, t_i *distance with respect to* ℓ*. This holds in both undirected and directed cases.*

Recall that in the single commodity case, there exists a feasible s-t flow of value at least d iff every s-t cut has capacity at least d.

One natural thing to try is to see if there is a cut condition for characterizing when there exists a multiflow.

Example 1.4. Let s_1, s_2, t_1, t_2 with $d_1 = 2$, $d_2 = 5$. Then suppose there is a cut $U \subseteq V$ with $s_1, s_2 \in U$ and $t_1, t_2 \notin U$. Then if there is a feasible multiflow, then the capacity of U must be at least 7. If A is a cut such that $s_1 \in A$ and $s_2, t_1, t_2 \notin U$, then the capacity of A is at least 2.

So the cut condition is necessary. We will see if it is also sufficient.

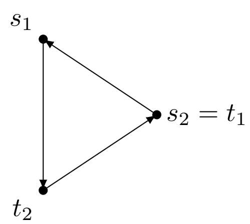
Definition 1.5.

1. For $U \subseteq V$, say (s_i, t_i) *crosses* U if $s_i \in U$, $t_i \notin U$.
2. The *cut condition* for U: the capacity of all arcs *leaving* U should be at least the demand of all (s_i, t_i) pairs that cross U.
3. An instance *satisfies the cut condition* if it satisfies the cut condition for every $U \subseteq V$, i.e., $\sum_{i : (s_i, t_i) \text{ crosses } U} d_i \leqslant c(\delta(U)) \; \forall U \subseteq V$.

We have seen that a feasible multiflow implies that the cut condition is satisfied.

It turns out that the reverse direction is not true: satisfying the cut condition is not sufficient for the feasibility of a multiflow.

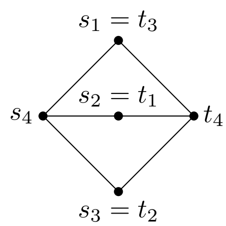
Example 1.6. In the directed case:



where $d_1 = d_2 = 1$, $c(e) = 1\ \forall e \in E$.

It is easy to verify that this satisfies the cut condition. But there is no f_1, f_2 that satisfy the demands, since there is only one path from s_1 to t_1 and one path from s_2 to t_2, but they overlap on an edge. More formally, it fails Onaga's Theorem for $\ell(e) = 1$.

Example 1.7. In the undirected case:



where $d_i = 1\ \forall i \in [4]$, $c(e) = 1\ \forall e \in E$.

This satisfies the cut condition. It also fails Onaga's Theorem for $\ell(e) = 1\ \forall e \in E$: $\mathrm{dist}_\ell(s_i, t_i) = 2\ \forall i \in [4]$, so $\sum_{i=1}^4 d_i \, \mathrm{dist}_\ell(s_i, t_i) = 8$, but $\sum_{e \in E} \ell(e)c(e) = 6$.

1.2 Sufficiency of the cut condition

The next natural question is, are there families of instances such that the cut condition is sufficient?

1.2.1 Directed Graphs

1. $k = 1 \implies$ the cut condition is sufficient
2. If $s_1 = \cdots = s_k$ or $t_1 = \cdots = t_k$, then the cut condition is sufficient.

It turns out that these are the only two cases where the cut condition is sufficient.

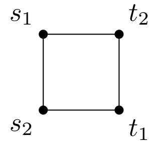
1.2.2 Undirected Graphs

There are more interesting cases, so this has been a rich area of study.

Theorem 1.8 (Hu)**.** *If $k = 2$, then satisfying the cut condition implies a feasible fractional multiflow.*

It does not, however, give a feasible integral multiflow.

Example 1.9.



where $d_1 = d_2 = 1$, $c(e) = 1$.

There is no integral multiflow, but there is a half-integral multiflow.

Theorem 1.10 (Hu)**.** *If $k = 2$, with integral demands and capacities, then if an instance satisfies the cut condition, there is a feasible half-integral multiflow.*

Lecture 29, December 8, 2015

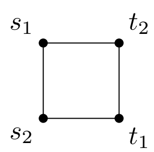
1 2-commodity multiflows in undirected graphs

Recall the k-commodity multiflow problem (in undirected graphs):
Definition 1.1. Given $G = (V, E)$ undirected, terminals $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ and demands $d_1, d_2, \ldots, d_k \in \mathbb{Q}_+$ and capacities $c : E \to \mathbb{Q}_+$, the *k-commodity multiflow problem* is to determine if there exists a feasible multiflow.

Recall:
Definition 1.2. An instance *satisfies the cut condition* if it satisfies the cut condition for every $U \subseteq V$, i.e., $\sum_{i:(s_i,t_i) \text{ crosses } U} d_i \leqslant c(\delta(U))\ \forall U \subseteq V$.

We have seen that a feasible multiflow implies that the cut condition is satisfied.
Theorem 1.3 (Hu)**.** *If $k = 2$, then satisfying the cut condition implies a feasible fractional multiflow.*

It does not, however, give a feasible integral multiflow.
Example 1.4.



where $d_1 = d_2 = 1$, $c(e) = 1$.

There is no integral multiflow, but there is a half-integral multiflow.
Theorem 1.5 (Hu)**.** *If $k = 2$, with integral demands and capacities, then if an instance satisfies the cut condition, there is a feasible half-integral multiflow.*

If we add extra conditions, we can in fact get an integral flow.
Definition 1.6. *Euler's condition*: $\forall v \in V$, $c(\delta(v)) + \sum_{i:s_i=v \text{ or } t_i=v} d_i$ is even.

It is easy to verify that Example 1.4 fails Euler's condition.
Theorem 1.7 (Rothschild-Whinston)**.** *If $k = 2$, with integral demands and capacities, then if an instance satisfies Euler's condition, then the cut condition implies a feasible integral multiflow.*

If we double all the demands and capacities in Example 1.4, then Euler's condition holds, and there is indeed an integral multiflow.

It is easy to see that Theorem 1.7 implies Theorem 1.3: since the demands are rational, we can scale up by a sufficiently large enough even integer until the demands and capacities are integral, and Euler's condition holds. Then we obtain an integral flow, which we can scale back down.

Before we prove Theorem 1.7, let us first prove a special case of Theorem 1.5:
Theorem 1.8. *Given $G = (V, E)$ undirected, $c(e) = 1\ \forall e \in E$, $(s_1, t_1), (s_2, t_2)$, $d_1 = d_2 = 1$. Then if the cut condition holds, then there exists a feasible half-integral multiflow.*

Proof. Add a supersource s connected to s_1 and s_2 using edges of capacity 1, and a supersink t connected to t_1 and t_2 using edges of capacity 1; call this graph G'.

Then if the cut condition holds in G, then any s-t cut in G' has at least two edges crossing it. So by Menger's theorem, there exist two edge-disjoint s-t paths. So there exist two edge-disjoint paths in P_1 and P_2 in G whose ends are $\{s_1, t_i\}$ and $\{s_1, t_{3-i}\}$. If $i = 1$ then we are already done.

So now suppose that $i = 2$, so that $P_1 : s_1 \rightsquigarrow t_2$ and $P_2 : s_2 \rightsquigarrow t_1$. If P_1 and P_2 are not vertex-disjoint, then we are also done, since we can just switch after the last vertex at which they intersect.

So suppose P_1 and P_2 are vertex-disjoint. If there are two edge-disjoint paths between P_1 and P_2, then if they are not vertex-disjoint, we have an integral multiflow; otherwise this is the same situation as Example 1.4, and we have a half-integral multiflow. So if we have two such paths, we are done. We need to argue that two such paths exist.

Attach a supersink u to all vertices in P_1 using edges of capacity 1, and a supersink v to all vertices in P_2 using edges of capacity 1; call this graph G''. Then since the cut condition holds in G, any u-v cut in G'' has at least two edges crossing it, so there exist two edge-disjoint paths from u to v, and thus two edge-disjoint paths Q_1 and Q_2 between vertices of P_1 and vertices of P_2. $\square$

We will now prove Theorem 1.7.

Proof of Theorem 1.7. Let us again add a supersource s connected to s_1 and s_2, and a supersink t connected to t_1 and t_2. But now the edge from s to s_1 and the edge from t_1 to t should have capacity d_1; and the edge from s to s_2 and the edge from t_2 to t should have capacity d_2. Again, call this G'.

If the cut condition holds in G, the capacity of all s-t cuts is at least $d_1 + d_2$. So there is an integral s-t flow g of value $d_1 + d_2$. The good condition is when all of the flow from s_1 ends up at t_1, and all of the flow from s_2 ends up at t_2.

Otherwise, let us construct G'' by adding u with an edge to s_1 of capacity d_1 and an edge to t_2 of capacity d_2, and v with an edge to s_2 of capacity d_2 and an edge to t_1 of capacity d_1. Then there exists an integral u-v flow h of value $d_1 + d_2$.

Claim 1.9. *We can assume that* $g(e) \equiv c(e) \bmod 2$*, and* $h(e) \equiv c(e) \bmod 2$*,* $\forall e \in E$*.*

We will come back to the proof of this claim, which is where we use Euler's condition.

Fix an arbitrary orientation of E, say $\vec{E}$. Define $g' : \vec{E} \to \mathbb{Z}$ as

$$g'(e) = \begin{cases} g(e) & \text{if } g \text{ uses } e \text{ in the forward direction} \\ -g(e) & \text{otherwise} \end{cases}$$

and define $h' : \vec{E} \to \mathbb{Z}$ similarly.

Note that $\forall e \in E$, $g'(e) \equiv c(e) \bmod 2$ and $|g'(e)| \leqslant c(e)$, and s_1 sends d_1 flow, s_2 sends d_2 flow, t_1 receives d_1 flow, and t_2 receives d_2 flow. Also, $\forall e \in E$, $h'(e) \equiv c(e) \bmod 2$ and $|h'(e)| \leqslant c(e)$, and s_1 sends d_1 flow, s_2 *receives* d_2 flow, t_1 receives d_1 flow, and t_2 *sends* d_2 flow.

Set $f_1 = \frac{1}{2}(g' + h')$, and $f_2 = \frac{1}{2}(g' - h')$. Since g' and h' have the same parity, f_1 and f_2 are integral. In f_1, s_1 sends d_1 flow, s_2 sends and receives nothing, t_1 receives d_1 flow, and s_2 sends and receives nothing. So f_1 is an integral s_1-t_1 flow of value d_1. One can similarly verify that f_2 is an integral s_2-t_2 flow of value d_2. It remains to verify that they satisfy the capacity constraints:

$$|f_1(e)| + |f_2(e)| = \frac{1}{2}(|g'(e) + h'(e)| + |g'(e) - h'(e)|) \leqslant \max\{|g'(e)|, |h'(e)|\} \leqslant c(e)\ \forall e \in E.$$

Proof of Claim 1.9. We will prove the claim for g. The same proof will hold for h.
We will show that $\forall v \in V$, $\{e \in \delta(v) : g(e) \not\equiv c(e) \bmod 2\}$ has even cardinality.

1. Let $v \in V - \{s_1, s_2, t_1, t_2\}$. Euler's condition gives $c(\delta(v)) \equiv 0 \bmod 2$, and by conservation of flow, $g(\delta(v)) = 0$. So $\sum_{e\in\delta(v)}(c(e) + g(e)) \equiv 0 \bmod 2$.
2. If $v = s_1$, then Euler's condition gives $c(\delta(s_1)) + d_1 \equiv 0 \bmod 2$, and $g(\delta(s_1)) = d_1$, so $\sum_{v\in\delta(s_1)}(c(e) + g(e)) \equiv 0 \bmod 2$. Similar for s_2, t_1, t_2.

So the graph $H = (V, E')$ where $E' = \{e \in E : g(e) \not\equiv c(e) \bmod 2\}$ is Eulerian. Pick a cycle C in H, and send one unit of flow in g along C. Then all edges in C have $g(e) \equiv c(e) \bmod 2$. We can repeat this until H is empty. ■

□

Theorem 1.7 can be used to show other properties, including a generalization of the max-flow-min-cut theorem that is known as the max-biflow-min-bicut theorem. This is still an active area of research.

1 Basics of Polyhedral Theory

We only state definitions and results here. Refer to [1] for proofs. Throughout this write-up we will restrict our focus to rational inputs. The reason being that the problem instances that arise in most practical applications are specified by rational values.

1.1 Cones, Polyhedra, Polytopes

Definition 1. 1. A set $S \subseteq \mathbb{R}^n$ is **convex** if $\lambda x + (1-\lambda)y \in S$ for every $x, y \in S, 0 \leq \lambda \leq 1$.

2. The **convex hull** of a set $S \subseteq \mathbb{R}^n$ is

$$\text{convex-hull}(S) := \left\{ \sum_{i=1}^{k} \lambda_i X_i : k \geq 1, finite, X_1, \ldots, X_k \in S, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_+, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

3. For a set of points $a_1, \ldots, a_k \in \mathbb{R}^n$, the combination $\sum_{i=1}^k \lambda_i a_i$ is

   (a) a **linear combination** if $\lambda_i \in \mathbb{R} \ \forall \ i \in [k]$,

   (b) an **affine combination** if $\lambda_i \in \mathbb{R} \ \forall \ i \in [k]$ and $\sum_{i=1}^k \lambda_i = 1$,

   (c) a **conic combination** if $\lambda_i \geq 0 \ \forall \ i \in [k]$,

   (d) a **convex combination** if $\lambda_i \geq 0 \ \forall \ i \in [k]$ and $\sum_{i=1}^k \lambda_i = 1$.

   The corresponding set of all linear/affine/conic/convex combinations of a finite number of points in a set $S \in \mathbb{R}^n$ is the **linear-hull**(S)**/affine-hull**(S)**/cone**(S)**/convex-hull**(S).

4. A **halfspace** in $\mathbb{R}^n$ is a set $\{x \in \mathbb{R}^n : a^T x \leq \delta\}$ for some $a \in \mathbb{R}^n, \delta \in \mathbb{R}$.
   A **hyperplane** in $\mathbb{R}^n$ is a set $\{x \in \mathbb{R}^n : a^T x = \delta\}$ for some $a \in \mathbb{R}^n, \delta \in \mathbb{R}$.

5. A **polyhedron** is the intersection of finitely many halfspaces, i.e., a set of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ for a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$.

6. A **polytope** is a bounded polyhedron.

7. A **cone** $C \subseteq \mathbb{R}^n$ is a set closed under conic-combination.

8. A **polyhedral cone** is a set of the form $\{x \in \mathbb{R}^n : Ax \leq 0\}$ for a matrix $A \in \mathbb{R}^{m \times n}$.

Theorem 2 (**Polyhedral cones $\equiv$ finitely generated cones**). *A polyhedral cone is generated by a conic combination of a finite set of vectors $a_1, \ldots, a_k$. The conic combination of a finite set of vectors $a_1, \ldots, a_k$ is a polyhedral cone.*

Definition 3. For two sets $S, T \subseteq \mathbb{R}^n$, the (Minkowski) sum is $S + T := \{s + t : s \in S, t \in T\}$.

Theorem 4 (**Polyhedral Decomposition**). *A polyhedron $P = \{x : Ax \leq b\}$ can be written as the Minkowski sum of a polytope Q and a polyhedral cone C (i.e., given a polyhedron P, there exists a polytope Q and a polyhedral cone C such that $P = Q + C$). Moreover, the cone C is the set $\{x : Ax \leq 0\}$ and is known as the* **characteristic cone of** *P.*

1.2 System of Inequalities

When does a system of linear inequalities have a feasible solution? In the following, let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. First recall when a system of linear equations has a feasible solution.

Theorem 5 (**Feasibility of a system of linear equalities**). *The following are equivalent:*

1. *The system $Ax = b$ has a solution $x \in \mathbb{R}^n$.*
2. *$rank(A) = rank([A|b])$.*
3. *b lies in the linear subspace spanned by the columns of A.*
4. *$\nexists\, y \in \mathbb{R}^m$ s.t. $y^T A = 0$, $y^T b \neq 0$.*

We have the following theorem characterizing the feasibility of a system of linear inequalities.

Theorem 6 (**Theorem of alternatives**). *Exactly one of the following two alternatives hold:*

1. *$\exists x \in \mathbb{R}^n$ s.t. $Ax \leq b$.*
2. *$\exists y \in \mathbb{R}^m$ s.t. $y \geq 0, y^T A = 0$ and $y^T b < 0$.*

A consequence of the theorem of alternatives is Farkas lemma which helps us determine when a system of linear inequalities has a feasible solution.

Theorem 7 (**Farkas Lemma**). *$Ax = b, x \geq 0$ is infeasible iff $\exists y : y^T A \geq 0,\ y^T b < 0$.*

Note that several variants of Farkas lemma exist. All of them can be shown by appropriately rewriting the matrix and the RHS vector and using the theorem of alternatives.

The theorem of alternatives/Farkas lemma has a geometric interpretation. It tells us that either b lies in the cone generated by the columns of the matrix A or there exists a hyperplane separating b from the cone generated by the columns of A. In fact if b lies in the cone generated by the columns of the matrix A then we can say more:

Theorem 8 (**Carathéodary's Theorem**). *Let $b \in Cone(S)$, where $S = \{a_1, \ldots, a_m\} \subseteq \mathbb{R}^n$. Then $b \in Cone(S')$ for some $S' \subseteq S$, where the vectors in S' are linearly independent. In particular, $|S'| \leq n$.*

1.3 Linear Programming

A linear programming problem is the optimization problem of minimizing/maximizing a linear function over a polyhedron. In the following, let $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$.

$$\begin{aligned} \max c^T x \text{ such that} \\ Ax \leq b. \end{aligned} \tag{P}$$

A vector $\overline{x}$ satisfying $A\overline{x} \leq b$ is said to be a feasible vector. If $\nexists x$ satisfying $Ax \leq b$, then the problem is said to be infeasible. If the objective value can be arbitrarily large, then the problem is said to be unbounded, otherwise it is bounded. An inequality $w^T x \leq \delta$ is called tight for a solution $\overline{x}$ if $w^T \overline{x} = \delta$. The problem (P) is known as the primal problem. The dual problem is built to get an upper bound on the value of the primal problem. The dual problem is given by

$$\begin{aligned} \min y^T b \text{ such that} \\ y^T A = c^T, \\ y \geq 0. \end{aligned} \tag{D}$$

Theorem 9 (**Weak Duality**). *For all feasible solutions x to (P) and y to (D), we have $c^T x \leq y^T b$.*

A consequence of weak duality is that if the primal is unbounded, then the dual has to be infeasible.

Theorem 10 (**Strong Duality**). *Suppose (P) and (D) are feasible. Then*

$$\max\{c^T x : Ax \leq b\} = \min\{y^T b : y \geq 0, y^T A = c^T\}.$$

Theorem 11 (**Complementary Slackness**). *Let x and y be feasible to (P) and (D) respectively. x and y are optimum to the respective problems iff $\forall i \in [m]$, either $y_i = 0$ or $\sum_{j=1}^{n} A_{ij} x_j = b_i$.*

1.4 Structure of Polyhedra

In this section, let $P = \{x : Ax \leq b\}$ be a polyhedron. We begin with a definition of the dimension of a polyhedra.

Definition 12.
1. A set of vectors $a_1 \ldots, a_k$ are **linearly independent** if $\sum_{i=1}^{k} \lambda_i a_i = 0$ implies $\lambda_i = 0 \ \forall \ i \in [k]$.
2. A set of vectors $a_1 \ldots, a_k$ are **affinely independent** if $\sum_{i=1}^{k} \lambda_i a_i = 0$ and $\sum_{i=1}^{k} \lambda_i = 0$ implies $\lambda_i = 0 \ \forall \ i \in [k]$.

Note that affine independence is invariant under translation of vectors.

Theorem 13 (**Linear vs Affine Independence**). *$a_1, \ldots, a_k \in \mathbb{R}^n$ are affinely independent iff*

$$\begin{bmatrix} a_1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} a_k \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1} \text{ are linearly independent.}$$

Definition 14. The **dimension of a polyhedron** P, denoted $\dim(P)$, is one less than the maximum number of affinely independent points in P.

The dimension of a polyhedron is given exactly by the dimension of the affine subspace of P (Theorem 17).

Definition 15. An inequality $a_i x \leq b_i$ in $Ax \leq b$ is an **implicit equality** if $a_i x = b_i$ for every $x \in P$.

Partition the set of inequalities $Ax \leq b$ defining P into two sets: $A^= x \leq b^=$ be the subsystem of implicit equalities and $A^+ x \leq b^+$ be the rest. The following theorem shows that P lies in the affine subspace defined by $A^= x = b^=$.

Theorem 16 (**Affine hull of a polyhedron**).

$$\textit{affine-hull}(P) = \{x : A^= x = b^=\} = \{x : A^= x \leq b^=\}.$$

Theorem 17 (**Dimension of a polyhedron**). *$dim(P) = dim(affine\text{-}hull(P)) = n - rank(A^=)$.*

Definition 18.
1. An inequality $a_i x \leq b_i$ of the system $Ax \leq b$ is a **redundant inequality** if the polyhedron $P = \{x : Ax \leq b\}$ is unchanged by removing it from the system.
2. An inequality $w^T x \leq \delta$ is a **valid inequality** for P if $w^T x \leq \delta \ \forall x \in P$.
3. A set $\{x : c^T x = \delta\}$ is a **supporting hyperplane** of $P = \{x : Ax \leq b\}$ if $\delta = \max\{c^T x : Ax \leq b\}$ and $c \neq 0$.

4. A set $F \subseteq P$ is a **face** of P if

   (a) either $F = P$,

   (b) or F is the intersection of P and a supporting hyperplane of P.

Note that a face F of a polyhedron is also a polyhedron. The following theorem gives a characterization of a face: A face is obtained by setting some inequalities to equalities in the description of P.

Theorem 19 (**Characterization of faces**). *A set F is a face of $P = \{x : Ax \leq b\}$ iff $F \neq \emptyset$ and $F = \{x \in P : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

Corollary 20. *1. The number of faces of a polyhedron $P = \{x : Ax \leq b\}$ is at most 2^m.*

2. Let F be a face of P and $F' \subseteq F$. Then F' is a face of P iff F' is a face of F.

3. The intersection of two faces of a polyhedron P is either a face of P or empty.

Definition 21. A **facet** is a maximal face distinct from P.

Theorem 22 (**Characterization of facets**). *Suppose no inequality in $A^+x \leq b^+$ is redundant in $Ax \leq b$. Then there exists a one-to-one correspondence between the facets of P and the inequalities in $A^+x \leq b^+$: each facet F of P is obtained as $F = \{x \in P : a_i^T x = b_i\}$ for an inequality $a_i^T x \leq b_i$ in $A^+x \leq b^+$.*

As a consequence if F is a facet of P, then $\dim(F) = \dim(P) - 1$.

Corollary 23. *1. Each face is the intersection of some facets of P.*

2. A polyhedron P has no facet iff P is an affine subspace.

Theorem 24 (**Inclusionwise minimal faces**). *Let $F \neq \emptyset$ be a face of P. It is an inclusionwise minimal face iff it is an affine subspace, i.e., $F = \{x : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

Definition 25. 1. A face is a **vertex** if it consists of a single point.

2. A face of dimension one is called an **edge**.

3. A point $x \in P$ is an **extreme point** of P if it cannot be expressed as a convex combination of two distinct points in P.

Theorem 26 (**Characterization of extreme points**). *The following are equivalent:*

1. $\overline{x}$ is an extreme point of P.

2. If $A'x \leq b'$ is the subsystem of $Ax \leq b$ satisfied as equality by $\overline{x}$, then rank$(A') = n$.

3. $F = \{\overline{x}\}$ is a face of P with dim$(F) = 0$.

4. $\exists c \in \mathbb{R}^n$ s.t. $\overline{x}$ is the unique optimum of the linear program $\max\{c^T x : x \in P\}$.

Corollary 27. *The vertices of a polyhedron P are precisely the extreme points of P.*

Corollary 28. *Suppose a polyhedron P has a vertex. If the linear program $\max\{c^T x : x \in P\}$ has an optimum solution, then it has an optimum solution x which is also an extreme point of P.*

Theorem 29 (**Characterization of polytopes**). *If P is a polytope, then $P =$ convex-hull(S), where S is the set of extreme points of P.*

All of the above material have been instrumental in the design and analysis of efficient algorithms to solve the linear programming problem.

References

[1] A. Shrijver, *Theory of Linear and Integer Programming*, Wiley, 1998.