Topology

MATH-GA 2310 and MATH-GA 2320

Sylvain Cappell

Transcribed by Patrick Lin

Figures transcribed by Ben Kraines

Abstract. These notes are from a two-semester introductory sequence in Topology at the graduate level, as offered in the Fall 2013–Spring 2014 school year at the Courant Institute of Mathematical Sciences, a school of New York University. The primary lecturer for the course was Sylvain Cappell. Three lectures were given by Edward Miller during the Fall semester.

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Parts I and II were covered in MATH-GA 2310 Topology I; and Parts III and IV were covered in MATH-GA 2320 Topology II.

The notes were transcribed live (with minor modifications) in LaTeX by Patrick Lin. Ben Kraines provided the diagrams from his notes for the course.

These notes are in a draft state, and thus there are likely many errors and inconsistencies. These are corrected as they are found.

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CHAPTER 0

Introduction

Before we begin to go through the theorems and proofs, we will go through, as an introduction, some problems and examples of Topology.

We will start with the idea of a space. We won't define a topological space yet, but for now we can look at some examples:

EXAMPLE 0.1. *Euclidean spaces* $X \subset \mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$. We have the usual notion of Euclidean distance between two points in the space:

$$d\left((x_1, \ldots, x_n), (y_1, \ldots, y_n)\right) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

EXAMPLE 0.2. The *n-dimensional disk* $D^n = \{v \in \mathbb{R}^n \mid \|v\| \leqslant 1\}$. Similarly, the n-dimensional *open disk* $\mathring{D}^n = \{v \in \mathbb{R}^n \mid \|v\| < 1\}$, and the $(n-1)$-dimensional *sphere* $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$.

REMARK 0.3. S^0 is comprised of the points ± 1 on a line.

We will also use the idea of continuity; we won't define this just yet, but for now we can perhaps talk about continuity using the ideas of Advanced Calculus.

We now look at some classic questions in Topology.

This is a classical questions that arises in many areas of mathematics:

QUESTION 0.4. Given a function $f : X \to X$, a point $u \in X$ is called a *fixed point* of f if $f(u) = u$. What can we say about whether or not f has fixed points?

EXAMPLE 0.5. If we have $v \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x) = x + v$, we see that f has no fixed points if $v \neq 0$. On the other hand, if $v = 0$, then every x is fixed. This is an extreme example.

EXAMPLE 0.6. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a rotation around the origin. Then there is exactly one fixed point: the origin.

Many other questions can, using a bit of cleverness, be turned into a question about having fixed points. As a result, there is a vast amount of literature dealing with this question. These results have had enormous impacts in many fields of mathematics.

We will look at some famous examples of spaces and fixed points.

THEOREM 0.7 (Brouwer Fixed-Point Theorem). *For any continuous function $f : D^n \to D^n$ there exists a fixed point.*

That is a very strong and remarkably general statement. The analogue fails for Euclidean space, for open disks and for spheres.

Example 0.8. For open disks, push everything to the right, but dampen the effect as we get closer and closer to the right side. As a concrete example, on the open interval $D^1 = (-1, 1)$ take $f(x) = (x + 1)/2$.

Example 0.9. For $X = S^n$, consider $\alpha(v) = -v$. This is the antipodal map.

Remark 0.10. Obviously the antipodal map does not work for $X = D^n$, as the origin remains fixed.

However, there is good theory (by Hadamard) about which functions $f : S^n \to S^n$ must have fixed points.

We will now attempt a sketch of the proof for the Brouwer Fixed-Point Theorem. We will get stuck, because we have not yet developed some of the machinery needed yet (that we will later in the course). But we will reach a point that is hopefully somewhat intuitive.

Proof-Sketch. We will use a proof by contradiction. Let $f : D^n \to D^n$. Assume that for all $x \in D^n$, $f(x) \neq x$. Draw the line from $f(x)$ through x to the edge S^{n-1}. Call $g(x)$ the point where the line meets S^{n-1}. Thus we have defined the function $g : D^n \to S^{n-1}$.

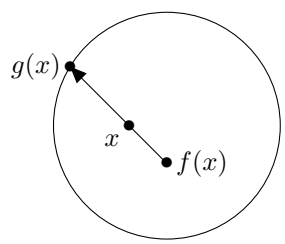

Figure 0.11.

It is not hard to check that g is continuous. Note that if $x \in S^{n-1}$, then $x = g(x)$, so g does not move points on S^{n-1}.

Consider the special case of $n = 1$. Then g sends values from $[-1, 1]$ to $\{\pm 1\}$, contradicts the Intermediate Value Theorem in Calculus, so g cannot be continuous.

For $n > 1$, the story gets more complicated; we need further methods. As a way of looking at it intuitively, though, consider what the function g does: since it pulls the disk to its edge, it will need to rip it somewhere; as a result, g must be discontinuous somewhere. To show this rigorously, we will need to develop ways of measuring "holes" in n dimensions. □

Remark 0.12. This does not show how to find the fixed point (this is a consequence of argument by contradiction).

Remark 0.13. Instead of focusing on f, we turned our attention to g. This is a nice feature of this argument: we turned from a question about points to one about the existence of continous maps between spaces.

We won't get to this for weeks, but a basic strategy in Algebraic Topology is to turn questions about spaces and continuous maps between them into problems in Algebra, such as problems about groups/vector spaces and homomorphisms/linear maps.

Another classic problem in Topology is one of classification:

QUESTION 0.14. Can we come up with a theory of classification for spaces?

Well, there are so many spaces that we can't really classify them all, but we would very much like to classify all "nice" spaces, for example *manifolds*. These are spaces which are everywhere locally modeled on the same Euclidean space $\mathbb{R}^n$.

EXAMPLE 0.15. S^2, which everywhere locally looks like $\mathbb{R}^2$.

REMARK 0.16. The only 1-dimensional manifolds are S^1 and D^1, and combinations thereof.

EXAMPLE 0.17. An example of a 2-dimensional manifold is the torus (or donut), which has one hole.

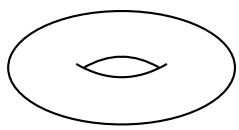

FIGURE 0.18.

Surfaces similar to the torus but with more holes are also 2-dimensional manifolds. This is an infinite family of 2-dimensional manifolds. In general, the number of holes is called the *genus*, denoted by g.

EXAMPLE 0.19. $\mathbb{R}\mathrm{P}^2$, the real projective space of dimension 2, which cannot be constructed in three dimensions (but can be in four). It can be constructed as follows: take D^2, which has S^1 as its edge, and the Möbius band, which also has S^1 as its edge. Since they have the same edge, we just glue them together along their edge. This yields $\mathbb{R}\mathrm{P}^2$. There are many other constructions of $\mathbb{R}\mathrm{P}^2$.

This can be generalized to an infinite family of manifolds in higher dimensions, for example, using D^3 and the Klein bottle.

A third classic question is as follows:

QUESTION 0.20. Given an n-dimensional manifold M^n, what is the smallest N such that M^n can be *embedded* in $\mathbb{R}^N$ (notated $M^n \hookrightarrow \mathbb{R}^N$)?

Happily, we know the following upper bound on N:

THEOREM 0.21 (Whitney Embedding Theorem). *Every n-dimensional manifold can be embedded into $\mathbb{R}^{2n}$.*

In fact, it is not hard to prove that $M^n \hookrightarrow \mathbb{R}^{2n+1}$; it takes a bit of work to get it down to $2n$. So the general question is somewhat closed, but it is still an interesting question ask when given a specific manifold, what is the smallest N that it fits into?

There are also deep connections between Topology and Analysis.

EXAMPLE 0.22. Consider a manifold, and a vector field on the manifold. Must it be zero somewhere?

THEOREM 0.23 (Poincaré-Bendixson Theorem). *Every manifold on S^{2k} has a point at which the vector field is zero.*

This is normally a problem in Ordinary Differential Equations, but it is also a problem in Topology.

Part I

Point-Set Topology

CHAPTER 1

Topological Spaces

In order to talk about most things about topological spaces, we need to set down a lot of basic things: we want to have things with a structure that is concrete.

1.1. Sets and Functions

We start by reviewing some basic notions of sets and functions.

Recall some examples of basic and familiar sets:

Example 1.1.1. The natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, etc.

Recall the basic notions of intersection, union, subtraction, distributivity and DeMorgan's Laws, and products.

Definition 1.1.2. A *function*, written $f : A \to B$ or $A \xrightarrow{f} B$, means that we associate each element $a \in A$ with some $f(a) \in B$. More formally, we can describe f using a relation $S = \{(a, f(a)) \mid a \in A\} \hookrightarrow A \times B$.

We have composition of function. We can write these as $A \xrightarrow{f} B \xrightarrow{g} C$, or $(g \circ f)(a) = g(f(a))$.

The concepts of injectivity and surjectivity are key. Again, let $f : A \to B$.

Definition 1.1.3. f is *injective* if $f(a_1) = f(a_2) \implies a_1 = a_2$. f is *surjective* if for each $b \in B$, there is some $a \in A$ such that $b = f(a)$. If f is both injective and surjective, we say that f is *bijective* and then we can talk about inverses, $f^{-1} : B \to A$, such that $f^{-1}(f(a)) = a$.

The bijections from a set A to itself form a group, called the Permutation Group. Bijections preserve structure of sets; functions that preserve structure of topologies are called homeomorphisms. We will get to those later.

We also have the notions of cardinality. If $f : A \to B$ is a bijection, then we say that A and B have the same cardinality, which we denote by $|A| = |B|$.

Example 1.1.4. The countable sets all have the same cardinality: $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{N}^2| = |\mathbb{Q}|$. We have the uncountable sets, that have the same cardinality as $\mathbb{R}$.

Exercise 1.1.5 (Cantor Diangolization Trick). Prove that $|\mathbb{N}| \neq |\mathbb{R}|$.

We have the notion of a *power set* of a set A, denoted as 2^A, which is the set of all subsets of A. The cardinality of 2^A is strictly larger than the cardinality of A.

Exercise 1.1.6. Prove that $|A| \neq |2^A|$.

Exercise 1.1.7 (Schroeder-Bernstein Theorem). Let $f : A \to B$ and $g : B \to A$ be injective. Prove that there is then some $h : A \to B$ that is a bijection.

Hint: First, as a special case, try $A = B = [0, 1]$, and set $f(x) = x/3$ and $g(x) = x$. Find h such that for some x, $h(x) = f(x)$, and for the rest, $h(x) = g^{-1}(x)$.

1.2. Topological Spaces

Definition 1.2.1. A *topological space* or *topology* is an ordered pair (X, O) where X is a set and O is a set of subsets of X, called *open sets*, satisfying three rules:

(1) $\emptyset, X$ are open.
(2) If $\{U_\alpha \mid \alpha \in \Lambda\}$ are open sets, then $\bigcup_\alpha U\alpha$ is open.
(3) If U, V are open, then $U \cap V$ are open.

When the context is clear, we will refer to X as the topological space and drop O from our notation.

Definition 1.2.2. A set $F \subset X$ is *closed* if $X - F$ is open.

Example 1.2.3. Consider the set $\mathbb{R}$ where open sets are unions of open intervals (a, b). Clearly, $\emptyset, \mathbb{R}$ are open, so condition (1) is satisfied. Furthermore, it is obvious that condition (2) is satisfied. It remains to show that the intersection of open sets are open, that is $\left(\bigcup_\alpha (a_\alpha, b_\alpha)\right) \cap \left(\bigcup_\beta (c_\beta, d_\beta)\right)$ is open.

Definition 1.2.4. A *basis* B for a topology (X, O) is a set of open sets so that:

(1) Every open set in O is a union of elements of B
(2) If $U_1, U_2 \in B$, then $U_1 \cap U_2 = \bigcup_k U_k$ where $U_k \in B$ for all k.

A topology can be generated by multiple bases.
We call open sets generated by B the unions of elements of B.

Example 1.2.5 (Trivial (or Indiscrete) Topology). For X, the only open sets are $\emptyset$ and X.

Example 1.2.6 (Discrete Topology). For X, let all subsets of X be open.

Example 1.2.7. For $\mathbb{C}$, let the open sets be $\emptyset$, $\mathbb{C}$, and $\mathbb{C} - A$ where A is the set of zeroes of some polynomial in one variable with complex coefficients.

Example 1.2.8 (Zariski Topology). A generalization of the previous: for $\mathbb{C}^n$, let the open sets be $\emptyset$, $\mathbb{C}^n$, and $\mathbb{C}^n - A$ where A is the set of common solutions to a system of polynomials in n variables and with complex coefficients.

Definition 1.2.9. The *interior* of a set A, $\mathrm{Int}(A)$, is the union of all open sets $U \subset A$. The *closure* $\overline{A}$ is the intersection of all closed sets $F \supset A$. We can also reformulate $\mathrm{Int}(A)$ as $\mathrm{Int}(A) = X - \overline{(X - \overline{A})}$. The *boundary* ∂A is $\overline{A} - \mathrm{Int}(A)$.

1.3. Metric Spaces

Definition 1.3.1. A function $d : X \times X \to \mathbb{R}$ is called a *metric* if for $x, y, z \in X$:

(1) $d(x, y) \geqslant 0$ and $d(x, y) = 0 \iff x = y$
(2) $d(x, y) = d(y, x)$
(3) (Triangle Inequality) $d(x, z) \leqslant d(x, y) + d(x, z)$

A space with a metric is a *metric space*.

EXAMPLE 1.3.2. The Euclidean spaces $\mathbb{R}$ with $d(x_1, x_2) = |x_1 - x_2|$ and the generalization $\mathbb{R}^n$ with $d(\vec{x}, \vec{y}) = \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2}$.

EXAMPLE 1.3.3 (An exotic example of a metric space: the p-adic numbers). For some prime p, we can write each integer m as $m = p^L q$ where p, q are relatively prime. Define $v_p(m) = L$. Then we can define a metric $d(m, n) = p^{-[v_p(m-n)]}$.

DEFINITION 1.3.4. An *open disk* around a point x is $\mathring{D}_r(x) = \{y | d(x, y) < R\}$. The *Metric Topology* is the one where open sets are the unions of open disks.

PROPOSITION 1.3.5. *The intersection of two open disks is a union of open disks.*

PROOF. Let $\mathring{D}_a(R)$ and $\mathring{D}_b(S)$ be two open disks, and z be in their intersection. Take $D_{\min\{R-d(a,z), S-d(b,z)\}}(z)$, which, by the triangle inequality, is a subset of $\mathring{D}_a(R) \cap \mathring{D}_b(S)$ by the triangle inequality. Hence around every point point in their intersection, we can find an open disk, so the intersection is a union of open disks. $\square$

REMARK 1.3.6. Only finite intersections of open disks are open. For example, in $\mathbb{R}$, take $\bigcap_{n=1}^{\infty}(-\frac{1}{2^n}, \frac{1}{2^n}) = \{0\}$, which is closed.

1.4. Constructing Topologies from Existing Ones

We now discuss some standard ways of construcing new topological spaces from existing ones.

A space can induce a topology onto its subsets.

DEFINITION 1.4.1. Let X be a space. If we have $A \subset X$, then the *subspace topology* of A is defined so that the open sets of A are the the sets $A \cap U$ where U is open in X.

We can also construct topological spaces using equivalence relations.

DEFINITION 1.4.2. Let X be a space with an equivalence relation $\sim$. Then take the map $\pi : X \to (X/\sim)$ sending $a \mapsto [a]$. Then the *quotient topology* on $X/\sim$ is defined so that the open sets of $X/\sim$ are the sets $[U]$ where the sets $\pi^{-1}(U)$ are open.

EXAMPLE 1.4.3. Consider the equivalence relation $\sim$ over $X = [0, 1]^2$ such that $(x, y) \sim (x, y)$ for $(x, y) \in (0, 1) \times [0, 1]$ and $(0, t) \sim (1, 1 - t)$. Then $X/\sim$ is the Mobius strip.

EXAMPLE 1.4.4. Given two copies of a space X, X_1 and X_2, then take $(X_1 \cup X_2)/\sim$ where for $u \in X$, if we call the copies of u in X_1 and X_2 by u_1 and u_2, respectively, then $u_1 \sim u_2$. Then $(X_1 \cup X_2)/\sim = X$.

We construct the topology of the product of two spaces as follows:

DEFINITION 1.4.5. Let X, Y be topological spaces. Then the *product topology* of $X \times Y$ is defined so that the open sets $U \times V \subset X \times Y$ are the ones where $U \subset X$ and $V \subset Y$ are open. Similarly, we can define the product topology on a finite product $X_1 \times \ldots \times X_n$.

EXAMPLE 1.4.6. The cone on X, $C(X)$, is obtained by $(X \times I)/(X \times \{1\})$. It is equivalent if we quotient by $X \times \{0\}$. The cone on S^{n-1} is D^n.

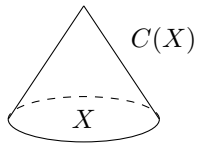

FIGURE 1.4.7.

EXAMPLE 1.4.8. We can describe the punctured plane as a product: $\mathbb{R}^2 \setminus \{0\} = S^1 \times \mathbb{R}^+$, using polar coordinates.

EXERCISE 1.4.9. Describe $\mathbb{R}^n \setminus \{0\}$ as a product space.

EXERCISE 1.4.10. Describe $S^n \setminus \{2 \text{ points}\}$ as a product.

Although some of these definitions may seem somewhat arbitrary, they are motivated by the concept of continuity: they are useful constructions such that the functions mapping the existing spaces to the new spaces are continuous. We will visit continuity shortly.

We now talk about a fourth way of constructing topologies from existing ones: gluing.

DEFINITION 1.4.11. *Gluing* two spaces is done as follows: given two spaces X, Y; some subset $A \hookrightarrow X$; and a continuous function $f : A \to Y$, let the set $X \cup_A Y$ be obtained from $(X \sqcup Y)/\sim$ where $\sim$ is given by $u \sim u$ for $u \in X - A$, $v \sim v$ for $v \in Y - f(A)$, and $w \sim f(w)$ for $w \in A$. Then $X \cup_A Y$ is a space that inherits its topology from X and Y.

EXAMPLE 1.4.12. Take two n-dimensional disks D_1^n and D_2^n. Glue them together along their edges S^{n-1}. The result is $D_1^n \cup_{S^{n-1}} D_2^n = S^n$, the n-dimensional sphere.

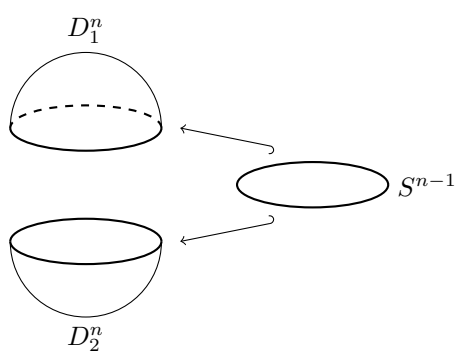

FIGURE 1.4.13.

As an alternate construction, let S_1^{n-1} and S_2^{n-1} denote the two edges of the cylinder (or equatorial band) $S^{n-1} \times I$ where I is some interval, usually the standard interval $[0, 1]$. Then $D_1^n \cup_{S_1^{n-1}} (S^{n-1} \times I) \cup_{S_2^{n-1}} D_2^n = S^n$.

Example 1.4.14. We can define the real projective space $\mathbb{R}\mathrm{P}^1$ in the following equivalent ways:

(1) The set of all lines through the origin in $\mathbb{R}^2$
(2) $[0,1]/\sim$ where $0 \sim 1$
(3) The circle S^1

In a similar mannar, we can define $\mathbb{R}\mathrm{P}^2$ as follows:

(1) The set of all lines through the origin in $\mathbb{R}^3$
(2) $S^2/\sim$ where $u \sim -u$ for $u \in S^2$
(3) $D^2/\sim$ where $u \sim -u$ for $u \in S^1 \hookrightarrow D^2$
(4) Take D^2 and the Mobius strip M. Note that $\partial D^2 = \partial M = S^1$. Then glue them together in the obvious manner along S^1 to obtain $D^2 \cup_{S^1} M$.

The last construction can be hard to visualize. We can equivalently do the same thing by taking $S^2 = D_1^2 \cup_{S_1^1} (S^1 \times I) \cup_{S_2^1} D_2^2$, and we can get $D^2 \cup_{S^1} M = S^2/\sim$ where $u \sim -u$ for $u \in S^2$, since $(D_1^2 \cup D_2^2)/\sim = D^2$ and $(S^1 \times I)/\sim = M$.

Note that since we can shrink D^2 to a single point, we can equivalently view the construction as M/S^1.

All of these constructions have obvious analogues in higher dimensions.

Exercise 1.4.15. Give an example where X, Y are disjoint spaces, $A \hookrightarrow X$, $f, g : A \to Y$ are homeomorphisms into their images, but $X \cup_{A,f} Y$ and $X \cup_{A,g} Y$ are not homeomorphic.

Example 1.4.16. The *suspension* of X, $\Sigma(X)$ is given by gluing two cones of X along their edges. This is equivalent to $(X \times I)/\sim$ where $(x_1, 0) \sim (x_2, 0)$ and $(x_1, 1) \sim (x_2, 1)$ for $x_1, x_2 \in X$.

Note that $\Sigma(S^{-1}) = S^0$.

Suspension is useful because it moves the dimension up, and many problems in topology get simpler as the dimension gets higher.

Exercise 1.4.17. Determine what $\Sigma(S^n)$ is. Furthermore, determine $\Sigma^k(S^n)$ for any k, including 0.

CHAPTER 2

Properties of Topological Spaces

2.1. Continuity and Compactness

Definition 2.1.1. Let $f : X \to Y$ be a function between two topological spaces. f is *continuous* if for each open set $U \subset Y$, then $f^{-1}(U) = \{x \mid f(x) \in U\}$ is open.

The following is a basic fact:

Theorem 2.1.2. *Let $f : X \to Y$ be a function between two metric spaces. Then the following are equivalent:*

(1) f is continuous.
(2) For every $x \in X$ and some $\varepsilon > 0$, then there is some $\delta > 0$, $\delta > 0$ such that $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$.

The proof is not difficult; it only requires unwinding the definitions of open sets and open disks in metric spaces.

Proof. Suppose f is continuous. Then for $a \in X$ and $\varepsilon > 0$, $\mathring{D}_\varepsilon(f(a))$ is open in Y. Since f is continuous, $f^{-1}[\mathring{D}_\varepsilon(f(a))]$ is open in X. Since open sets in X are unions of open disks, $a \in f^{-1}[\mathring{D}_\varepsilon(f(a))]$ can be found in some open disk $\mathring{D}_R(b) \subset f^{-1}[\mathring{D}_\varepsilon(f(a))]$. Since $\mathring{D}_{R-d(a,b)}(a) \subset \mathring{D}_R(b) \subset f^{-1}[\mathring{D}_\varepsilon(f(a))]$, setting $\delta = R - d(a, b)$ gives us (1).

In the other direction, let V be open in Y, and $a \in f^{-1}(V)$. Since V is open there is some open disk $\mathring{D}_{\varepsilon_a}(f(a)) \subset V$. Then there is some $\delta_a > 0$ such that $\mathring{D}_{\delta_a}(a) \subset f^{-1}[\mathring{D}_\varepsilon(f(a))]$. Then since $f^{-1}(V)$ is contained within $\bigcup_{a \in f^{-1}(V)} \mathring{D}_{\delta_a}(a)$ and vice versa, $f^{-1}(V)$ is open. $\square$

Definition 2.1.3. A bijection $f : X \to Y$ between two topological spaces is a *homeomorphism* if f is a bijection of open sets, that is, f and f^{-1} are both continuous.

We will now talk about a very important concept in Topology: compactness.

Definition 2.1.4. Let $\{U_\alpha\}$ be a set of open sets. We say $\{U_\alpha\}$ is an *open cover* of a set A if $A \subset \bigcup_\alpha U_\alpha$. A *finite subcover* is a finite subset of $\{U_\alpha\}$ that is still an open cover of A.

Definition 2.1.5. A topological space X is *compact* if for each open covering of X, there exists a finite subcover.

The following theorem is a major motivation for the notion of compactness, as the properties outlined make many things, like integration, simple.

Theorem 2.1.6. *If $f : X \to \mathbb{R}$ is continous and X is compact, then*

(1) f is bounded.
(2) f achieves its maximum and minimum.
(3) If (X, d) is a metric space with the metric topology then f is uniformly continuous.
(4) Each sequence $x_1, x_2, \ldots$ has a convergent subsequence

$$x_{i_1}, x_{i_2}, \ldots, x_{i_L}, i_1 < i_2 < \cdots < i_L,$$

i.e. there is some $p \in X$ such that

$$\lim_{j\to\infty} d(x_{i_j}, p) = 0.$$

Proof. (1) Note that $\mathbb{R} = \bigcup_{i=1}^{\infty}(-i, i)$. Since f is continuous, then $X = \bigcup_{i=1}^{\infty} f^{-1}(-i, i)$. But since X is compact, it can be covered by finitely many sets $(-i, i)$. Then there is some i_α such that $-i_\alpha \leqslant f(x) \leqslant i_\alpha$ for $x \in X$. So f is bounded.

(2) Since f is bounded, then for some $K > 0$ such that $-K \leqslant f(x) \leqslant K$. Take the least upper bound (LUB) of $f(X)$. If the LUB is not achived, then $\bigcup_{i=1}^{\infty} U_i = (-\infty, \text{LUB} - 1/2^i)$ contains $f(X)$, so $\{f^{-1}(U_i)\}$ is an open cover of X. Since X is compact, there is a finite subcover, so for some finite N we have $f(X) \subset U_N$, a contradiction since $N <$ LUB. A similar argument holds for the greatest lower bound.

(3) For $\varepsilon > 0$, we need $\delta > 0$ so that $d(x_1, x_2) < \delta \implies d(f(x_1), f(x_2)) < \varepsilon$. For continuous $f : X \to Y$ with their respective metrics, and X compact, for each $a \in X$ set δ_a so that $d(a, x) < d_a \implies d(f(a), f(x)) < \varepsilon/2$. The open disks $\{\mathring{D}_{\delta_a/2}(a)\}$ form an open cover of X; since X is compact there is a finite subcover of open disks around N points $\{a_1, \ldots, a_N\}$. Define $\delta = \frac{1}{2}\min\{\delta_{a_1}, \ldots, \delta_{a_N}\}$. Let $x_1, x_2 \in X$ where $d(x_1, x_2) < \delta$. From the finite subcover we can find some open disk $D_{\delta_{a_J}/2}(a_J)$ so that $d(x_1, a_J) < \delta_{a_J}/2$. We also have $d(x_1, x_2) < \delta \leqslant \delta_{a_J}/2$, so we have $d(a_J, x_2) < \delta_{a_J}$. Since x_1, x_2 are both found in $D_{\delta_{a_J}}(a_J)$, we have $d(f(x_1), f(a_J)) < \varepsilon/2$ and $d(f(a_J), f(x_2)) < \varepsilon/2$, so $d(f(x_1), f(x_2)) < \varepsilon$.

(4) If X is a finite set it is obvious. Otherwise, for sake of contradiction assume instead that there is no convergent subsequence. Then let $S = \{x_i\}$, and $p \in X - S$. We can find a disk D of radius $\varepsilon(p)$ sufficiently small such that $D_{\varepsilon(p)}(p) \cap S = \emptyset$. Otherwise if $p \in S$ then there are finitely many i with $x_i = p$. We can find a disk D of radius $\varepsilon(p)$ sufficiently small such that $D_{\varepsilon(p)}(p) \cap S = \{p\}$. Consider the open cover of X by $\{\mathring{D}_{\varepsilon(p)}(p) \mid p \in X$. Since X is compact, there is a finite subcover $\{\mathring{D}_{\varepsilon(p_j)}(p_j) \mid j = 1, \ldots, N\}$, so the union of these finite open disks cover all of S, but each contains only a finite number of points in S, a contradiction. □

Theorem 2.1.7. *The following facts are also known about compactness:*

(1) If $f : X \to Y$ is continuous, then X is compact implies that Y is compact.
(2) If X and Y are compact, then $X \times Y$ is compact.
(3) If $[-K, K]$ are compact, then $[-K, K]^n$ is compact.
(4) $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Exercise 2.1.8. Prove Fact 2.1.7(2).

Exercise 2.1.9. Prove that a closed subspace of a compact space is compact.

2.2. Hausdorff Spaces

PROPOSITION 2.2.1. *A is a closed and bounded subset of $\mathbb{R}^n$ if and only if A with the subspace topology is compact.*

DEFINITION 2.2.2. A space X is *Hausdorff* if for $x, y \in X$ there exist open $U_x, U_y \subset X$ such that $x \in U_x, y \in U_y$, and $U_x \cap U_y = \emptyset$.

EXAMPLE 2.2.3. Metric spaces are Hausdorff.

LEMMA 2.2.4. *If $A \subset X$ and X is Hausdorff, then A being compact implies that A is closed.*

2.3. Connectedness

We will discuss two notions of connectedness: connectedness and the stronger condition of path connectedness.

DEFINITION 2.3.1. A space X is said to be *connected* if for any decomposition of X into two open sets, $X = A \cup B$ where $A \cap B = \emptyset$, then either $A = X$ or $B = X$ and the other is empty.

EXERCISE 2.3.2. Show that if $f : X \to Y$ is a continuous surjective map between spaces, then X is connected implies Y is connected.

Before we talk about path connectedness, we need the definition of a path. These will always be described using a parametrization.

DEFINITION 2.3.3. A *path* in a space X is a continuous function $\omega : I \to X$. $p = \omega(0)$ is called the *initial point* and $q = \omega(1)$ is the *terminal point*. These are the *end points* of ω, and ω is a path from p to q.

REMARK 2.3.4. If there is a path from p to q then it is obvious that there is a path from q to p.

DEFINITION 2.3.5. The *inverse path* is $\omega^{-1}(t) = \omega(1-t)$ for $0 \leqslant t \leqslant 1$.

A useful thing we can do is splice two paths together.

DEFINITION 2.3.6. Given a path α from p to q and path β from q to r, we can obtain the *product path* $\alpha \cdot \beta$ from p to r given by

$$\alpha \cdot \beta = \begin{cases} \alpha(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}.$$

REMARK 2.3.7. Note that this operation is not associative, because the parameterization is different. For example, if we have paths α from p to q, β from q to r, and γ from r to s, then

$$(\alpha \cdot \beta) \cdot \gamma = \begin{cases} \alpha(4t) & 0 \leqslant t \leqslant \frac{1}{4} \\ \beta(4t-1) & \frac{1}{4} \leqslant t \leqslant \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \quad \text{whereas} \quad \alpha \cdot (\beta \cdot \gamma) = \begin{cases} \alpha(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(4t-2) & \frac{1}{2} \leqslant t \leqslant \frac{3}{4} \\ \gamma(4t-3) & \frac{3}{4} \leqslant t \leqslant 1 \end{cases}$$

We can define the trivial path at $p \in X$ by $e_p(t) = p$ for $0 \leqslant t \leqslant 1$, but by the previous remark, this does not act as an identity element.

DEFINITION 2.3.8. A space X is called *path connected* if given any two points $p, q \in X$ there is a path from p to q.

Example 2.3.9. We know that I is connected.

Exercise 2.3.10. Show that if a space is path connected, then it is connected.

Example 2.3.11 (A space that is connected but not path connected). Take the Y axis unioned with the graph $\{(t, \sin \frac{1}{t}) \mid t > 0\}$. Take a point p on the graph and a point q on the Y axis, there is no path from p to q since it would need to go through infinitely many cycles of $\sin \frac{1}{t}$. However, the space is connected.

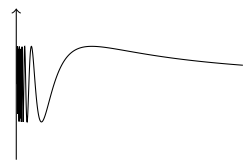

Figure 2.3.12.

The concept of multiplication of paths by itself is not so exciting. We could work with it even if it does not have commutativity, but it does not even have associativity. So we need something better.

Part II

The Fundamental Group

CHAPTER 3

Basic Notions of Homotopy

3.1. Homotopy of Paths

We will now discuss the concept of homotopy of paths. Define a relation, called homotopy, between two paths with given endpoints. The idea is a parametrized deformation of paths without moving the endpoints.

Example 3.1.1. If we have two paths α and β from p to q around half of a torus, they are homotopic, but if we have another path γ also from p to q but going around the other half, γ is not homotopic to α and β because we cannot deform it through the "hole" in the torus.

Definition 3.1.2. A *homotopy* of paths α and β from p to q in X is a continuous function $H : I \times I \to X$ parametrized by (t, s) with $H(t, 0) = \alpha(t)$, $H(t, 1) = \beta(t)$, $H(0, s) = p$, and $H(1, s) = q$. α and β are *homotopic*, written $\alpha \underset{h}{\sim} \beta$, if there is a homotopy of α and β.

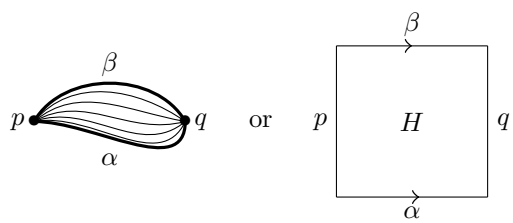

Figure 3.1.3.

This is an equivalence relation among paths from p to q. If $\alpha \underset{h}{\sim} \beta$ and $\beta \underset{h}{\sim} \gamma$, then $\alpha \underset{h}{\sim} \gamma$.

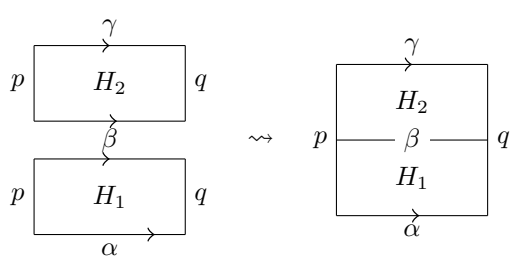

Figure 3.1.4.

Exercise 3.1.5. If $\alpha \underset{h}{\sim} \beta$ and $\beta \underset{h}{\sim} \gamma$, give an explicit reparametrization to show that $\alpha \underset{h}{\sim} \gamma$.

The notion of homotopy is compatible with the idea of path multiplication. So if we have two paths $\alpha \underset{h}{\sim} \beta$ from p to q and $\gamma \underset{h}{\sim} \delta$ from q to r, we have $\alpha \cdot \gamma \underset{h}{\sim} \beta \cdot \delta$.

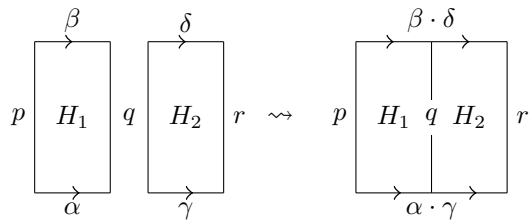

Figure 3.1.6.

Exercise 3.1.7. Given two paths $\alpha \underset{h}{\sim} \beta$ from p to q and $\gamma \underset{h}{\sim} \delta$ from q to r, give an explicit reparametrization to show that $\alpha \cdot \gamma \underset{h}{\sim} \beta \cdot \delta$.

In fact, using homotopy we obtain associativity for path multiplication:

Proposition 3.1.8. *Given paths α from p to q, β from q to r, and γ from r to s, we have $(\alpha \cdot \beta) \cdot \gamma \underset{h}{\sim} \alpha \cdot (\beta \cdot \gamma)$.*

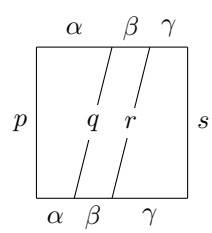

Figure 3.1.9.

Exercise 3.1.10. Give an explicit parametrization to show $(\alpha \cdot \beta) \cdot \gamma \underset{h}{\sim} \alpha \cdot (\beta \cdot \gamma)$.

Recall the trivial path $e_p(t) = p$ for $0 \leqslant t \leqslant 1$. We noted that this does not function under normal equality as an identity element due to different parametrization. On the other hand, under homotopy, the trivial path does indeed act as an identity element.

Exercise 3.1.11. Given a path α from p to q, write out a homotopy in terms of alpha showing $\alpha \cdot e_q \underset{h}{\sim} \alpha$.

Remark 3.1.12. We can show similarly that $e_p \cdot \alpha \underset{h}{\sim} \alpha$.

Example 3.1.13. If we go from p to q and back along the same path, this is not the same as having not having moved at all. Formally, for α from p to q, $\alpha \cdot \alpha^{-1} \neq e_p$. However, $\alpha \cdot \alpha^{-1} \underset{h}{\sim} e_p$.

Exercise 3.1.14. Given α from p to q, write out a homotopy in terms of alpha showing $\alpha \cdot \alpha^{-1} \underset{h}{\sim} e_p$.

Unfortunately, since multiplication cannot always be performed between paths, we do not yet have a group! However, if we restrict our attention to only loops, we can achieve a group.

3.2. The Fundamental Group

Pick a point $x \in X$. We call this the basepoint. A loop based at x is a path whose endpoints are both x. We will denote by $[\alpha]$ the homotopy class of α.

Definition 3.2.1. Let $\pi_1(X, x) = \{\text{loops based at } x\}/\underset{h}{\sim}$, the set of all loops based at x up to homotopy.

Since all loops based at x can be multiplied together and we have shown associativity, identity, and inverse, we arrive at the following:

Proposition 3.2.2. *$\pi_1(X, x)$ is a group.*

We call $\pi_1(X, x)$ the *fundamental group* of X. It turns out to also be the first in a whole family of groups, so it is also called the *first homotopy group*. Historically, it has also been called the *Poincaré group*, which is no longer used since his name is attached to many other groups as well.

The following is an interesting result on homotopy and the fundamental group.

Recall that X is convex if given $u, v \in X \subset \mathbb{R}^n$ then $(tu + (1-t)v) \in X$ for $0 \leqslant t \leqslant 1$.

Proposition 3.2.3. *If $X \subset \mathbb{R}^n$ is convex, then for $p, q \in X$ and any paths α, β from p to q are homotopic.*

Proof. We simply linearly interpolate: $H(t, s) = (1-s)\alpha(t) + s\beta(t)$. □

Corollary 3.2.4. *If X is convex, then the fundamental group is given by $\pi_1(X, x) = \{[e_x]\}$, which we also notate as 0.*

Definition 3.2.5. If X is a path connected space with $\pi_1(X, x) = 0$, we say that X is *simply connected*.

Intuitively, this means that there are no one-dimensional holes. We will explain this in more detail shortly. Note that while convexity implies simply connectedness, the converse does not hold.

There are several ways to think about loops based at $x \in X$:

(1) A path $I \xrightarrow{\omega} X$ where $\omega(0) = \omega(1) = x$.
(2) A path $S^1 \xrightarrow{\gamma} X$ with $\gamma(1, 0) = x$. This is essentially the same as the previous since $S^1 = I/(0 \sim 1)$.
(3) A map $\mathbb{R} \xrightarrow{\delta} X$ with $\delta(0) = x$ and $\delta(\mathbb{R} \setminus \text{compact set}) = x$.

Using these notions, we can get the following:

Proposition 3.2.6. *To say that $[\alpha] = e$ in $\pi_1(X, x)$ is equivalent to saying α regarded as a map $S^1 \to X$ extends to a continuous map $D^2 \to X$. In other words, the loop α can be "filled in".*

Proof. Assume $[a] = e$. We are given some homotopy H that can be considered as a unit square where the left, bottom, and right sides all map to the basepoint x. Then if we quotient the unit square by these three sides, we get the disk D^2.

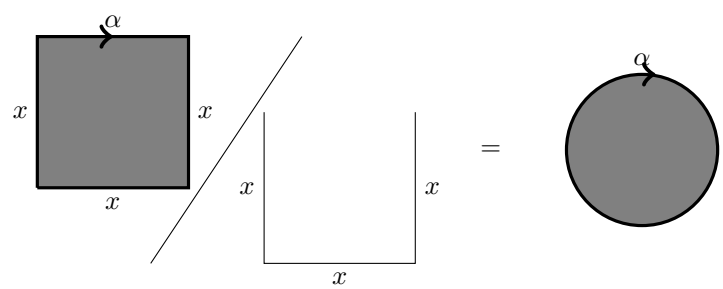

Figure 3.2.7.

In the other direction, using the mapping $q : I \times I \to (I \times I)/\text{three sides} = D^2$ and the map $L : D^2 \to X$ such that $L\,|_{S^1} = \alpha$, we get the map $L \circ q$ is a homotopy of α to the trivial map. □

Corollary 3.2.8. *To say that $\pi_1(X, x) = 0$ is equivalent to saying that any mapping $S^1 \xrightarrow{\gamma} X$ can be extended to D^2.*

Remark 3.2.9. Any sphere is simply connected because any path on it can be filled in, but a torus is not because a loop around hole cannot.

The following proposition is a very powerful one.

Proposition 3.2.10 (Independence of Basepoint). *If X is a path connected space and $p, q \in X$, then $\pi_1(X, p) \cong \pi_1(X, q)$.*

Proof. Pick a path ω from p to q. Now define $\Phi_\omega : \pi_1(X, p) \to \pi_1(X, q)$ by $\Phi_\omega([\alpha]) = [\omega^{-1}\alpha\omega] \in \pi_1(X, q)$.

We first show that Φ_ω is well-defined. If $[\alpha] = [\beta]$ in $\pi_1(X, p)$ then we have $\omega^{-1}\alpha\omega \underset{h}{\sim} \omega^{-1}\beta\omega$, so that $[\omega^{-1}\alpha\omega] = [\omega^{-1}\beta\omega]$ in $\pi_1(X, q)$.

Next, we check that Φ_ω is homeomorphic. That is, $\Phi_\omega([\alpha][\gamma]) = \Phi_\omega([\alpha])\Phi_\omega([\gamma])$ in $\pi_1(X, q)$. Well, the LHS is $\omega^{-1}\alpha\gamma\omega$ and the RHS is $(\omega^{-1}\alpha\omega)(\omega^{-1}\gamma\omega)$. Since homotopy preserves associativity and inverse, we can cancel to see that they are equal.

In a similar fashion, we can use $\Phi_{\omega^{-1}}$ the other way.

The last thing to do is to check that $\Phi_{\omega^{-1}} = \Phi_\omega^{-1}$. Take $\Phi_{\omega^{-1}} \circ \Phi_\omega([\alpha])$. This is $[(\omega^{-1})^{-1}(\omega^{-1}\alpha\omega)\omega^{-1}]$, which is just $[\alpha]$. The opposite direction is similar. □

This is not completely satisfying. There are many isomorphisms between spaces. But we often want a natural isomorphism. However, we have not shown that there is a natural isomorphism from $\pi_1(X, p)$ to $\pi_1(X, q)$.

Remark 3.2.11. The choice of isomorphism from $\pi_1(X, p)$ to $\pi_1(X, q)$ may well depend on the choice of the path ω from p to q.

3.3. Excursion: Basic Notions of Group Theory

We will now review some useful group theory.

DEFINITION 3.3.1. A *homomorphism* φ between two groups G, H is a map such that $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$ for $\alpha, \beta \in G$.

DEFINITION 3.3.2. By the *kernel* of G we mean $\mathrm{Ker}(\varphi) = \{g \in G | \varphi(g) = e\}$.

EXERCISE 3.3.3. Show that $\mathrm{Ker}(\varphi)$ is a subgroup of G.

DEFINITION 3.3.4. A homomorphism φ is an *isomorphism* if φ is also a bijection. Then we write $G \cong H$. This is an equivalence relation among groups.

EXAMPLE 3.3.5. The group $\mathbb{Z}_2$ under addition is isomorphic to the set $\{\pm 1\}$ under multiplication.

DEFINITION 3.3.6. An isomorphism from G to itself is called an *automorphism* of G; the set of all automorphisms from G to itself $\mathrm{Aut}(G)$ is a group under composition.

EXAMPLE 3.3.7. The automorphism group of the integers $\mathrm{Aut}(\mathbb{Z})$ is $\{\pm 1\} \cong \mathbb{Z}_2$.

EXAMPLE 3.3.8. The automorphism of $\mathbb{Z}_p$ for p prime $\mathrm{Aut}(\mathbb{Z}_p)$ is $\mathbb{Z}_p^\times$, which is $\{\mathbb{Z}_p \setminus \{0\}$ under $\times\}$. A fundamental result in number theory is that $\mathbb{Z}_p^\times \cong \mathbb{Z}_{p-1}$.

DEFINITION 3.3.9. Given $g \in G$, consider $\rho_g(\alpha) = g^{-1}\alpha g$. This operation is called *conjugation by* g. This is an automorphism of G, and these are called the inner automorphisms of G, denoted by $\mathrm{Inn}(G)$.

EXERCISE 3.3.10. Show that G is abelian if and only if $\mathrm{Inn}(G) = \{\mathrm{Id}_G\}$.

A general remark about group isomorphisms: Given two isomorphisms $\alpha, \beta : A \to B$, we can view them as related in the following way:

$$\beta = \beta \circ (\alpha^{-1} \circ \alpha) = (\beta \circ \alpha^{-1}) \circ \alpha$$

But $\beta\alpha^{-1}$ is an automorphism of B, so they differ by an automorphism of B.

EXERCISE 3.3.11. Show that $\beta = \alpha \circ$ (automorphism of A).

Let us look again the Fundamental Group. Say we have two isomorphisms Φ_α, Φ_β from $\pi_1(X, p)$ to $\pi_1(X, q)$. Recall our definition that if γ is a loop based at p in X then $\Phi_\alpha[\gamma] = [\alpha^{-1}\gamma\alpha]$ and similarly, $\Phi_\beta[\gamma] = [\beta^{-1}\gamma\beta]$. Comparing Φ_α with Φ_β, we see that each is equal to the other composed with automorphisms. eg. $\Phi_{\beta^{-1}} \circ \Phi_\alpha$, but $\Phi_{\beta^{-1}} \circ \Phi_\alpha[\gamma] = [\beta\alpha^{-1}\gamma\alpha\beta^{-1}] = [\beta\alpha^{-1}][\gamma][\alpha\beta^{-1}]$. This is just the conjugate of $[\gamma]$ by $[\alpha\beta^{-1}] \in \pi_1(X, p)$. So Φ_β is Φ_α composed with some conjugation, so the isomorphisms are said to be "the same up to conjugation". This leads to the following:

PROPOSITION 3.3.12. *If we have isomorphisms Φ_α and Φ_β from $\pi_1(X, p)$ to $\pi_1(X, q)$, then one can be written as the other conjugated by some automorphism.*

COROLLARY 3.3.13. *If $\pi_1(X, p)$ is abelian, this isomorphism is independent of the choice of the path.*

This follows immediately from Exercise 3.3.10.

Exercise 3.3.14. Show that conversely, if α is a path from p to q, so that $\Phi_\alpha : \pi_1(X,p) \to \pi_1(X,q)$ then if Φ_α is conjugated with some automorphism, the result is equal to some isomorphism $\Phi_\beta : \pi_1(X,p) \to \pi_1(X,q)$ for some choice of β from p to q. In particular, whether or not a path-connected space X is simply connected is independent of the choice of basepoint.

3.4. Maps Between Spaces and Induced Homomorphisms

Before we begin let us set down some notation. The notation $X_{,x}$ means that we assume $x \in X$ to be the basepoint. We often assume $y = f(x)$, that is, f is "basepoint-preserving". We will also use the notation space$_+$ to mean a space with a basepoint, and map$_+$ to mean a map that preserves basepoint.

Let $f : X_{,x} \to Y_{,y}$ be a continuous map between spaces. We would like to study what this tells us about the relation between "holes" in X and "holes" in Y. To this end, observe that f determines ("induces") a corresponding homomorphism of the fundamental groups $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ given by $f_*([\gamma]) = [f \circ \gamma]$.

Proposition 3.4.1. *f_* is a homomorphism and is well-defined.*

Note that we need to check that f_* is well-defined since it is defined on equivalence relations.

Proof. Suppose we have two loops γ and δ such that $\gamma \underset{h}{\sim} \delta$. To show that f_* is well-defined, we need to show that $f \circ \gamma \underset{h}{\sim} f \circ \delta$. Well, if H is a homotopy from γ to δ, then $f \circ H$ is a homotopy from $f \circ \gamma$ to $f \circ \delta$. So f_* is well-defined. Next, it is obvious that $f_*([\alpha].[\beta]) = f_*([\alpha]).f_*([\beta])$, so f_* is indeed a homomorphism. $\square$

Example 3.4.2. If $f(x) = y$, that is, f is a constant map, then $f \circ \gamma$ is the constant loop at y, since $f_*([\gamma]) = [f \circ \gamma] = [e_Y]$. In other words, $f_* = 0$.

Example 3.4.3. If $f(x) = \mathrm{Id}_X(x)$, then $(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x)}$.

Proposition 3.4.4. *The construction of induced maps satisfies the following basic properties:*

1. *Given Id_X, this induces $(\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x)}$. This means that when we move from topology to algebra, we preserve the identity.*
2. *This is compatible with composition: Say $f : X_{,x} \to Y_{,y}$ and $g : Y_{,y} \to Z_{,z}$ are continuous maps$_+$. Then $g_* \circ f_* = (g \circ f)_*$.*

Proof. We already have the preservation of identity by Example 3.4.3. Next, $(g \circ f) \circ (\gamma) = (g \circ (f \circ \gamma))$, so $(g \circ f)_*([\gamma]) = g_*(f_*[\gamma])$. $\square$

This notion is often called *naturality*, but more properly, *functorality.*

Example 3.4.5. Suppose $f : X_{,x} \to Y_{,y}$ is a homeomorphism$_+$. Let $g = f^{-1}$, then $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. This induces $f_* : \pi_1(X,x) \to \pi_1(Y,y)$ and $g_* : \pi_1(Y,y) \to \pi_1(X,x)$, such that $g_* \circ f_* = (g \circ f)_* = (\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(X,x)}$, and similarly, $f_* \circ g_* = \mathrm{Id}_{\pi_1(Y,y)}$. So f_* and g_* are inverse isomorphisms of groups.

Example 3.4.6. Take $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. We will see later that $\pi_1(S^1) \cong \mathbb{Z}$. In analysis, this assigns each loop what is called a winding number. This is the number of times the loop goes around the circle. Let $g_k : S^1 \to S^1$ where $k \in \mathbb{Z}$, $k \geqslant 0$ given by $g_k(z) = z^k$. Then $(g_k)_* : \pi_1(S^1) \to \pi_1(S^1)$. Let $u = [\mathrm{Id}_{S^1}]$.

Then $(g_k)_*([u]) = [g_k]$. We can also view this as $\underbrace{u \circ \ldots \circ u}_{k \text{ times}} = u^k \in \pi_1(S^1)$, which corresponds with $k \in \mathbb{Z}$. So we can view this as $(g_k)_* : \mathbb{Z} \to \mathbb{Z}$ where $(g_k)_*(n) = kn$. By the naturality property, if we have some $(g_l)_*$, we have $(g_k)_* \circ (g_l)* = (g_{kl})_*$ is just multiplication by kl.

COROLLARY 3.4.7. *Homeomorphic spaces have isomorphic fundamental groups.*

The fundamental group captures information about the holes, and the induced maps captures what the maps are doing to the holes. We are throwing out so much rich information, but we can analyze this so much more easily. Unlike spaces that have uncountably many points, most of their fundamental groups are countable. So we will end up converting problems in geometry to problems in algebra.

3.5. Homotopy of Maps and Spaces

Often we will be faced with too many continuous maps between spaces. So just as we cut down the number of loops by imposing equivalences on them via homotopies, we will introduce an equivalence relation among the continuous maps from X to Y.

DEFINITION 3.5.1. If $f, g : X \to Y$ are continuous maps, a homotopy of f to g is a continuous map $H : I \times X \to Y$ with $H(0, u) = f(u)$ and $H(1, u) = g(u)$. If there is a homotopy from f to g we write $f \underset{h}{\sim} g$.

This is the notion of continuous deformations between maps.

$f \underset{h}{\sim} g$ is an equivalence relation among continuous maps from X to Y. Just as with homotopy of paths, we just need to reparameterize.

If the homotopy preserves basepoints we will write $f \underset{h}{\sim}_+ g$. This is again an equivalence relation.

PROPOSITION 3.5.2. *If $f \underset{h}{\sim}_+ g$, then f_* and g_* are equal as homomorphisms $\pi_1(X, x) \to \pi_1(Y, y)$.*

PROOF. Let γ be a loop in X. Define $\Gamma(t, v) = (t, \gamma(v)) \in I \times X$ for $v \in X$. Say H is a homotopy from f to g, then $H \circ \Gamma$ is a homotopy of loops $f \circ \gamma$ to $g \circ \gamma$. Well, $[f \circ \gamma] = [g \circ \gamma]$ in $\pi_1(Y, y)$ so $g_*([\gamma]) = f_*([\gamma])$ and $f_* = g_*$ are homomorphisms. $\square$

EXERCISE 3.5.3. Show that if Y is convex in $\mathbb{R}^n$, any two maps $f, g : X \to Y$ are homotopic.

EXAMPLE 3.5.4. Let $f : S^1 \to S^1$ loop around three times then back once. Then f_* is multiplication by two.

Let us set down some more notation. $\{X, Y\}$ is sometimes used to denote the set of continuous maps from X to Y. In fact, we can give a natural topology on this space, in which homotopy of maps desribes a path in this space. $[X, Y]$ is often used to denote $\{X, Y\}/\underset{h}{\sim}$. We will use $[X, Y]_+$ to denote the subset that preserves basepoint.

EXAMPLE 3.5.5. $[S^1, Y_{,y}]_+ = \pi_1(Y, y)$.

In fact, in general, $[S^k, Y_{,y}]_+ = \pi_k(Y, y)$. It takes some work to show that these are still groups.

The notion of homotopy equivalence underlies a large part of topology. Recall that if $f : X \to Y$ is a homeomorphism and $g = f^{-1}$, then $f \circ g = \mathrm{Id}_X$ and $g \circ f = \mathrm{Id}_Y$. But we can live with a weaker notion than equality: for the fundamental group we can just require homotopy. This leads to the following:

Definition 3.5.6. $f : X \to Y$ is a *homotopy equivalence* if there is a map $g : Y \to X$ such that $g \circ f \underset{h}{\sim} \mathrm{Id}_X$ and $f \circ g \underset{h}{\sim} \mathrm{Id}_Y$. Then we say that X is homotopic equivalent to Y.

This notion can easily be extended to preserve basepoints.

The following is an easy consequence of the previous discussion.

Proposition 3.5.7. *If f is a basepoint-preserving homotopy equivalence, then $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism.*

Exercise 3.5.8. Prove Proposition 3.5.7.

Hint: Apply naturality and that homotopic maps have the same induced homomorphisms.

Example 3.5.9. $\mathbb{R}^n$ is not homeomorphic to a point, but $\mathbb{R}^n$ is homotopic equivalent to a point: take $f : \text{Point} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \text{Point}$. Then $g \circ f = \mathrm{Id}_{\text{Point}}$ and $f \circ g \underset{h}{\sim} \mathrm{Id}_{\mathbb{R}^n}$.

So homotopy equivalence is much weaker than homeomorphism since it can change dimension, but homotopy equivalence actually preserves the fundamental group. We will get to this later.

Example 3.5.10. S^1 is homotopic equivalent to $\mathbb{R}^2 \setminus (1 \text{ point})$. Similarly, we have $\mathbb{R}^2 \setminus (k \text{ points})$ is k loops joined together at a point.

Example 3.5.11. S^1 is homotopic equivalent to the annulus.

In fact, algebraic topology has trouble distinguishing between homotopic equivalent spaces, but it is often easy to distinguish between spaces that are not just by computing the fundamental groups.

Remark 3.5.12. A warning: if we are given an inclusion of spaces $A_{,a} \overset{i}{\hookrightarrow} X_{,x}$ then the induced map $i_* : \pi_1(A, a) \to \pi_1(X, a)$ need not be injective.

Example 3.5.13. Take $S^1 \overset{i}{\hookrightarrow} D^2$. Then $i_* : \pi_1(S^1) \to \pi_1(D^2)$ maps $\mathbb{Z} \to 0$.

3.6. Retractions

Definition 3.6.1. A subspace $A \overset{i}{\hookrightarrow} X$ is called a *retract* of X if there is a map $X \overset{r}{\to} A$ such that $r \circ i = \mathrm{Id}_A$. Then r is called a *retraction* (of X to A.

The idea is that r pulls X back to its subspace A.

Example 3.6.2. Take $S^1 \overset{i}{\hookrightarrow} \mathbb{R}^2 \setminus \{0\}$. Then $r : \mathbb{R}^2 \setminus \{0\} \to S^1$ given by $r(v) = \frac{v}{\|v\|}$ is a retraction from $\mathbb{R}^2 \setminus \{0\}$ to S^1. The idea is that everything outside gets shoved in and everything inside gets pushed out. We wouldn't know what to do with the origin, but happily that's not present.

Definition 3.6.3. Given two spaces $X_{,x}, Y_{,y}$, then $X \vee Y = (X \cup Y)/(x \sim y)$. This is obtained by gluing two spaces together at a point.

Example 3.6.4. Take $X \overset{i}{\hookrightarrow} X \vee Y$ and let y be the basepoint of Y. Then $r : X \vee Y \to X$ given by

$$r(u) = \begin{cases} y & u \in Y \\ u & u \in X \end{cases}$$

is a retraction. So X is a retract of $X \vee Y$ (and similarly, so is Y).

Let us look at what retractions mean for the fundamental group of a space. Let $A_{,a} \overset{i}{\hookrightarrow} X_{,a} \overset{r}{\to} A_{,a}$ where $r \circ i = \mathrm{Id}_A$. Then the induced maps are given by $\pi_1(A, a) \overset{i_*}{\hookrightarrow} \pi_1(X, a) \overset{r_*}{\to} \pi_1(A, a)$, where $(r \circ i)_* = r_* \circ i_* = (\mathrm{Id}_A)_* = \mathrm{Id}_{\pi_1(A)}$. So we get the same story in the induced map. The result of this is that i_* is injective, and so we can regard $\pi_1(A, a)$ as a subgroup of $\pi_1(X, a)$.

Example 3.6.5. Take the spaces and maps from Example 3.6.4. Then we conclude that $\pi_1(X) \overset{i_*}{\hookrightarrow} \pi_1(X \vee Y)$ (similarly, $\pi_1(Y)$ is a subset of $\pi_1(X \vee Y)$ as well).

So this gives us the result that the fundamental group can tell us about whether or not we can have a retraction between two spaces.

Example 3.6.6. We saw in Example 3.6.2 that $\mathbb{R}^2 \setminus \{0\}$ retracts to S^1. We now show that $\mathbb{R}^2$ does not. We can see this by taking the induced map i_* of the inclusion map from S^1 to $\mathbb{R}^2$. But since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(\mathbb{R}^2) = 0$, i_* is not an inclusion map.

Definition 3.6.7. Given an inclusion $A \overset{i}{\hookrightarrow} X$, A is called a *deformation retract* of X if there is a retraction $X \overset{r}{\to} A$ with $r \circ i = \mathrm{Id}_A$ and $i \circ r \underset{h}{\sim} \mathrm{Id}_X$. This is called a *deformation retraction*.

The idea is that the homotopy shrinks X down to its subspace A.

Example 3.6.8. Recall Example 3.6.2, with $r(v) = \frac{v}{\|v\|}$. Using the homotopy $H(v, t) = tv + (1 - t)\left(\frac{v}{\|v\|}\right)$, we see then that $H(\cdot, t) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ at any time t, with $H(v, 0) = \frac{v}{\|v\|}$ and $H(v, 1) = v$. Hence we have a deformation retraction.

So deformation retraction is kind of like a halfway point between homeomorphism and homotopy equivalence. In fact, the existence of a deformation retraction implies homotopy equivalence. In fact, in practice, this is often how homotopy equivalences arise! Recall that a homotopy equivalence of two spaces implies that they have the same fundamental group. So whereas in any retraction i_* is only injective, in a deformation retraction i_* is bijective.

Example 3.6.9. Take $S^1 \overset{i}{\hookrightarrow} M$ where M is the Mobius strip. This is a deformation retract, so S^1 is homotopy equivalent to M and so $\pi_1(M) \cong \pi_1(S^1) = \mathbb{Z}$.

Example 3.6.10. In Example 3.6.4 we saw that X is a retract of $X \vee Y$, but X is usually not a deformation of $X \vee Y$. We will show later that often $X \vee Y$ has a larger fundamental group that X.

The following notion is sometimes used:

Definition 3.6.11. An inclusion $A \overset{i}{\hookrightarrow} X$ is a *strong deformation retract* if there is a retract $X \overset{r}{\to} A$ with a homotopy $H : X \times I \to X$ with $H(u, 0) = u$, $H(u, 1) = (i \circ r)(u)$ and (the extra definition) $H(u, t) = u$ for $u \in A, t \in I$. That is, A never moves.

It takes some more technical details to show, but in fact if there is a deformation retract, we can modify it to obtain a strong deformation retract.

We will not talk much about strong deformation retracts.

Example 3.6.12. The retract from Example 3.6.2 is a strong deformation retract.

Exercise 3.6.13. Show that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

Example 3.6.14. The inclusion $\bigvee_k S^{n-1} \overset{i}{\hookrightarrow} \mathbb{R}^n \setminus \{k \text{ points}\}$ is a (strong) deformation retract.

Definition 3.6.15. A subspace $Y \subset \mathbb{R}^n$ is called *star-like* if there is a point $y \in Y$ such that for any point $u \in Y$, the straight line segment $\{tu + (1-t)y \mid t \in I\}$ from y to u is in Y.

This is weaker than convex, as for convexity this must be true for any $y \in Y$, not just a fixed one. So convexity implies star-like.

Exercise 3.6.16. Show the following:

(1) If Y is star-like in $\mathbb{R}^n$, any two maps $f, g : X \to Y$ are homotopic.
(2) If Y is star-like, then $\{y\} \hookrightarrow Y$ is a deformation retract.
(3) Conclude that Y is homotopic equivalent to a point.

Definition 3.6.17. A space is called *contractible* if it is homotopic equivalent to a point.

CHAPTER 4

Comparing Fundamental Groups of Spaces

In order to compare different spaces, we would like to compute the fundamental groups of various spaces.

4.1. Fundamental Groups of Product Spaces

PROPOSITION 4.1.1. *Given two spaces X, Y, the fundamental group of their product is the product of their fundamental groups, i.e.* $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

PROOF. Take the projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. We have the induced maps

$$(p_1)_* : \pi_1(X \times Y) \to \pi_1(X) \quad \text{and} \quad (p_2)_* : \pi_1(X \times Y) \to \pi_2(Y).$$

Then we claim that $(p_1)_* \times (p_2)_* : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ is an isomorphism.

To see that $(p_1)_* \times (p_2)_*$ is surjective, take $[\alpha] \in \pi_1(X), [\beta] \in \pi_1(Y)$. Then $[\alpha \times \beta] \in \pi_1(X \times Y)$ satisfies $(p_1)_* \times (p_2)_*([\alpha], [\beta]) = [\alpha] \times [\beta]$.

To see that $(p_1)_* \times (p_2)_*$ is injective, we just show that $\operatorname{Ker}((p_1)_* \times (p_2)_*) = \emptyset$. Suppose we have a loop $\gamma \in \operatorname{Ker}((p_1)_* \times (p_2)_*)$, we can write $\gamma = (\gamma_1, \gamma_2)$. So $\pi_1([\gamma]) = [\gamma_1] = e \in \pi_1(X)$ and similarly, $\pi_2([\gamma]) = [\gamma_2] = e \in \pi_2(Y)$. Then we have a homotopy H_1 in X of γ_1 to the constant loop. Similarly we have H_2 in Y of γ_2 to the constant loop. So we can combine these to get $H_1 \times H_2 : S^1 \times I \to X \times Y$, which is a homotopy of γ to the constant map. So $[\gamma]$ is trivial. $\square$

By induction, we can show that this works for products of even more spaces. We won't show this, but this actually works for infinite products as well.

EXAMPLE 4.1.2. A torus T is homeomorphic to $S^1 \times S^1$. Thus the fundamental group is $\pi_1(T) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z}^2$. Similarly, the fundamental group of a n-torus is $\mathbb{Z}^n$.

EXAMPLE 4.1.3. More generally, for any space X, $\pi_1(X \times S^1) = \pi_1(X) \times \mathbb{Z}$.

This result about products is nice because if the fundamental groups of two spaces are abelian, then so is the fundamental group of their product. Most fundamental groups are nonabelian, eg. the fundamental group of wedge of two spaces that are not simply connected is never abelian. Later we will prove that every group is the fundamental group of some space.

COROLLARY 4.1.4. *Any product of simply connected spaces is simply connected.*

EXAMPLE 4.1.5. Any product of spheres is simply connected.

This shows that as we go up in dimensions, we get more and more simply connected spaces.

4.2. Fundamental Group of Glued Spaces

Recall that another way of constructing spaces is by gluing, so let us look at the fundamental group of spaces glued together.

We will need some notions from group theory.

Definition 4.2.1. Given a group G, a set $S \subset G$ is said to *generate* G if $G = \{x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k} \mid x_k \in S, \alpha_k \in \mathbb{Z}\}$, where we allow repeats. Equivalently, if the smallest supergroup containing S is G. Then G is *generated* by S.

Example 4.2.2. $\mathbb{Z}^2$ is generated by $\{(1,0),(0,1)\}$.

Definition 4.2.3. A group is called *finitely generated* if it is generated by some finite set S.

Exercise 4.2.4. Show that $\mathbb{Q}$ under addition is not finitely generated.

Remark 4.2.5. Any finite group G is trivially finitely generated, taking $S = G$.

In this course we will concern ourselves mostly with finitely generates spaces, since for reasonable spaces, the fundamental group is generally finitely generated.

Consider the following example:

Proposition 4.2.6. *Let $X = A \cup B$ where A, B are open subsets of X, with a basepoint $x \in A \cap B$. Suppose $A, B, A \cap B$ are path connected. Then $\pi_1(X, x)$ is generated by $\pi_1(A, x) \cup \pi_1(B, x)$.*

We will see later that this becomes the first part of what is known as Van Kampen's Theorem.

Remark 4.2.7. A warning: we do need to assume that $A \cap B$ is path connected.

Example 4.2.8. Consider $X = S^1$ where A, B are each just over half of a loop, on opposite sides of the circle.

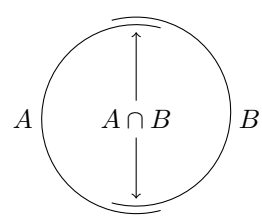

Figure 4.2.9.

Then $X = A \cup B$ and $\pi_1(A) = 0$ and $\pi_1(B) = 0$. The problem arises because $A \cap B$ is disconnected.

Corollary 4.2.10. *If A, B are simply connected and $A \cap B$ are path connected, then $X = A \cup B$ is also simply connected.*

Example 4.2.11. Take $X = S^n$ where $n > 1$. Decompose it into $X = A \cup B$ where A, B are each just over half of a hemisphere, on opposite ends of the sphere. Well, $A \approx \mathring{D}^n$ is simply connected, and so is B. So we conclude S^n is simply connected. Note that this works for $n > 1$ since the intersection $A \cap B$, the equatorial band, is in fact path connected.

COROLLARY 4.2.12. *If K is a set of generators for $\pi_1(A,x)$ and L is a set of generators for $\pi_1(B,x)$, then $K \cup L$ is a set of generators for $\pi_1(X,x)$ where $X = A \cup B$.*

EXAMPLE 4.2.13. Consider $\mathbb{R}\mathrm{P}^2$, which is Mobius $\cup_{S^1} D^2$. Well, since D^2 is simply connected $\pi_1(D^2) = \{e\}$. On the other hand, the Mobius strip is homotopy equivalent to S^1, so $\pi_1(\text{Mobius}) = \mathbb{Z}$. Let $w = [S^1]$ be a generator for $\mathbb{Z}$. We see that $\pi_1(\mathbb{R}\mathrm{P}^2)$ is generated by w. Notice that $w^2 = [\text{edge of Mobius}]$ which can be filled in by D^2, so $w^2 = e$ in $\pi_1(\mathbb{R}\mathrm{P}^2)$. In summary, $\pi_1(\mathbb{R}\mathrm{P}^2)$ is generated by w with $w^2 = e$. If $w = e$ then $\pi_1(\mathbb{R}\mathrm{P}^2) = 0$. This turns out not to be true, but we cannot prove this yet. The other possibility is that $\pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}_2$, which is correct.

REMARK 4.2.14. Here the Mobius strip and D^2 are closed sets. However, we can easily replace them by slighly larger open sets which are homotopy equivalent. Then since they have the same fundamental groups as their closed counterparts, we get the same conclusion.

PROOF OF PROPOSITION 4.2.6. Let $\gamma : I_{,0} \to X$ be a continuous function where $\gamma(0) = \gamma(1)$ with $A \subset X$ and $B \subset X$. Now consider $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$. These are open sets, so they are a union of intervals. Since I is compact, we can cover each with finitely many such intervals. Now if overlapping intervals are in $\gamma^{-1}(A)$, we can combine them to get fewer intervals. Then we can view $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ as unions of intervals that alternate. We can make these into closed subintervals by breaking the interval up choosing points in the overlaps. In other words, we can choose $0 = t_0 < \cdots < t_m = 1$ such that each $[t_i, t_{i+1}]$ lies in $\gamma^{-1}(A)$ or in $\gamma^{-1}(B)$. Hence for any loop in X we can split it into paths.

Now for each $\gamma(t_i)$, pick a path δ_i from $\gamma(t_i)$ to the basepoint $x \in A \cap B$. Denote $\sigma_i = \gamma([t_{i-1}, t_i]) \in A \cap B$. Then $\gamma = \sigma_1.\sigma_2.\ldots.\sigma_m$. Then we also have

$$\gamma \underset{h}{\sim} \sigma_1\delta_1\delta_1^{-1}\sigma_2\delta_2\delta_2^{-1}\ldots\sigma_n$$

where $\delta_i\delta_i^{-1}$ are loops! Hence we now get the result:

$$[\gamma] = [\sigma_1\delta_1][\delta_1^{-1}\sigma_2\delta_2][\delta_2^{-1}\sigma_3\delta_3]\ldots[\delta_{n-1}^{-}1\sigma_n],$$

where each piece $[\delta_{i-1}^{-1}\sigma_i\delta_i]$ is a loop in A or B. □

4.3. Excursion: Combinatorial Group Theory

To obtain the second part of Van Kampen's Theorem, we need yet more ideas from combinatorial group theory.

4.3.1. Free Groups. Let L be a set, called the *set of letters*, and $W(L)$ the set of all *words* in the letters and their inverses (including the empty word). The problem is that we can have the word $xyx^{-1}xy^{-1}yx$, where there are obvious cancellations we have not done. Hence we can define $\hat{W}(L)$ as the set of words without an adjacent letter with its inverse. Then we have the reduction map $r : W(L) \to \hat{W}(L)$ that cancels any adjacent letter with its inverse. We will skip over the details (as the proof can be found in any book on groups) that this map is well-defined.

DEFINITION 4.3.1. Define a multiplication among reduced words u and v by $uv = r(u \text{ followed by } v)$. This is the obvious multiplication, and this forms a group, called the *free group* on the set of letters L, written $F(L)$ or F_L.

EXAMPLE 4.3.2. If $L = \{l\}$, then $F(L) = \{\ell^k\} \cong \mathbb{Z}$. This is often written F_1.

EXAMPLE 4.3.3. If $L = \{a, b\}$ has two elements, then $F_2 = F(L)$ is much larger.

EXERCISE 4.3.4. In the free group of two generators F_2, how many words are there of length 2?

Well, there are at least 2^m strings of length m, but at most 4^m. So these are large groups, since they grow exponentially in size.

In fact, there is an entire field of study about classifying groups based on how quickly they grow in terms of length of string. Some grow polynomially, some grow exponentially. The free group is the fastest growing group.

Let us take a small excursion to linear algebra. We can describe a basis in two ways: the first is that all vectors in the space can be described as the sum of multiples of basis elements; the second is that if B is a basis of V, given any other vector space W and a function $\varphi : B \to W$, then there exists a unique linear map $\Phi : V \to W$ which is φ on the restriction to the basis B.

$$\begin{tikzcd} B \arrow[r, hook] \arrow[dr, "\varphi"'] & V \arrow[d, "\exists!"] \\ & W \end{tikzcd}$$

FIGURE 4.3.5.

This is defined in the obvious way, writing $v \in V$ as $v = a_1 b_1 + \ldots + a_k b_k$ where $B = \{b_1, \ldots, b_k\}$ then $\Phi(v) = a_1\varphi(b_1) + \ldots + a_k\varphi(b_k)$. (The minimality of B is forced by the uniqueness of Φ.)

A similar story exists in free groups, since the letters are like a basis and the words are like vectors. A free group $F(L)$ has the following analogous property: given any group W and a map $\varphi : L \to W$, then there exists a unique homomorphism $\Phi : F(L) \to W$ with Φ on the restriction to L is just φ.

$$\begin{tikzcd} L \arrow[r, hook] \arrow[dr, "\varphi"'] & F(L) \arrow[d, "\exists!"] \\ & W \end{tikzcd}$$

FIGURE 4.3.6.

Φ is defined as $\Phi(\ell_{i_1}^{\pm 1}\ell_{i_k}^{\pm 1} \ldots \ell_{i_k}^{\pm 1} = \varphi(\ell_{i_1})^{\pm 1}\varphi(\ell_{i_2})^{\pm 1} \ldots \varphi(\ell_{i_k})^{\pm 1}$ where these ℓ_i's are in L. We could have defined the free group this way and then showed that it is given by reduced words.

REMARK 4.3.7. There are two problems with this: it is not obvious that there is a group with this property, and it is not obvious that there is only one. The

first can be solved by using $F(L)$. The proof of uniqueness is as follows: suppose there were two, $F^1(L)$ and $F^2(L)$, but since $L \hookrightarrow F^1(L)$ and $L \hookrightarrow F^2(L)$ there is a unique homomorphism $\Phi_{1,2} : F^1(L) \to F^2(L)$ and similarly $\Phi_{2,1}$ going back. It remains to check that the composites are Id, but by the uniqueness of these maps this follows easily.

Algebraists often prefer this more abstract definition since for any algebraic structure there is some "free" structure that follows analogously.

THEOREM 4.3.8. *Every group is the quotient of a free group. In particular, every finitely generated group is the quotient of a free group generated on a finite set of letters.*

PROOF. Pick any set of generators K for a group G. Take $F(K)$. Take the map $\Phi : F(K) \to G$ that extends a map $\varphi : K \to G$. Well, Φ is a surjective funciton, and $G \cong F(K)/\operatorname{Ker}(\Phi)$. □

EXAMPLE 4.3.9. Using the free group $\mathbb{Z}$, we can get $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ by quotienting out by the even numbers.

However, to describe a group, it is not sufficient to just describe the generators; we need more information. We want something of the form $G = F/A$. We could describe A by picking generators for it, but this often yields a large and redundant set.

EXAMPLE 4.3.10. Suppose x, y are mapped to the non-trivial element, writing $\mathbb{Z}_2 = F(x, y)/A$ we have $A = \{x^2, y^2, xy, yx\}$. For bigger groups and more generators, this grows very quickly.

But we can take advantage of the fact that the kernel of any homomorphism is not just a subgroup, but it is a normal subgroup (i.e. it is closed under conjugation). So we don't need to list all of the generators! This leads to the following:

DEFINITION 4.3.11. A set of elements $C \subset G$ where G is a group is said to *normally generate* G if the elements of C and their conjugates (in G) generate G.

EXAMPLE 4.3.12. Given a finite group G, $G = F(S)/A$ where S is a finite set and A is a subgroup normally generated by a finite set. To show that this is true, use the multiplication table of G, which for $a, b \in G$ gives $c = ab \in G$, then $(ab)c^{-1} = e$. Then we can just write

$$G = F(G)/\{(ab)c^{-1} \mid a, b \in G, c = ab\}.$$

DEFINITION 4.3.13. A *presentation* of of a group G is $G = F/R$ where F is a free group and R is the subgroup normally generated by a collection of elements, called *relations*.

EXAMPLE 4.3.14. Take a clock with k hours. Then there are $2k$ symmetries, generated by a rotation σ and a flip τ. This is called D_{2k}, the dihedral group of order $2k$. We can write a presentation as

$$D_{2k} = \{\sigma, \tau \mid \sigma^k = \tau^2 = \tau\sigma\tau\sigma = e\}.$$

All other relations are consequences of these three, i.e. they are conjugates of products of these.

Definition 4.3.15. A group G is *finitely presented* if there is a presentation of G with finitely many generators and finitely many relations.

Note that if a set generates a group, then it normally generates the group, but not vice versa.

Example 4.3.16. Take the group $S_3 = P(a, b, c)$. Then $R = \{(a\ b)\}$ normally generates S_3.

Exercise 4.3.17. Prove that $R = \{(a\ b)\}$ normally generates S_3 but does not generate it.

Remark 4.3.18. Similar results can be similarly shown for any S_n.

A common notation for the presentation of a group is $G = \{x_1, \ldots, x_n \mid R\}$, which means $G = F(x_1, \ldots, x_n)/\langle R\rangle$ where $\langle R\rangle$ is the normal subgroup K generated by R and its conjugates.

Example 4.3.19. We can express $F_n = \{x_1, \ldots, x_n \mid \emptyset\}$. We can also write $F_n = \{x_1, \ldots, x_n, y \mid y\}$.

Such an addition of a generator and then killing it off is often called *stabilization* of presentation.

Example 4.3.20. $\mathbb{Z}_k = \{x \mid x^k\}$

Example 4.3.21. As an example of a group that is finitely generated but not finitely presented, start with $\mathbb{Z}^\infty = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$. This is the subgroup of $\mathbb{Z} \times \mathbb{Z} \times \ldots$ consisting of sequences with finitely many non-zero entries. This is generated by $x_1 = (1, 0, 0, \ldots), x_2 = (0, 1, 0, \ldots), x_3 = (0, 0, 1, \ldots), \ldots$, etc. Note that this group is not finitely generated. Consider

$$G = \{t, x_1, x_2, \ldots \mid [x_i, x_j] = e, tx_it^{-1} = x_{i+1}\}$$

where $[u, v] = uvu^{-1}v^{-1}$ is the commutator. Notice that since $t^kx_1t^{-k} = x_{k+1}$, G is actually just generated by $\{t, x_1\}$, but there are an infinite number of relations. Hence G is finitely generated but not finitely presented.

Often, when given a presentation of a group, it may be hard to figure out what group has been given.

Example 4.3.22. Take

$$G = \{x, y, z \mid yxy^{-1} = x^2, zyz^{-1} = y^2, xzx^{-1} = z^2\}.$$

If G were made abelian, then it would only consist of only the trivial element (G is called *perfect*). In fact, $G = \{e\}$, but this is not obvious! Take the same story on four elements,

$$L = \{w, x, y, z \mid xwx^{-1} = w^2, yxy^{-1} = x^2, zyz^{-1} = y^2, wzw^{-1} = z^2\}.$$

But Serre observed that L is, in fact, not only nontrivial, but not finite!

There are ways to modify relations. The *Tietze moves* on relations are as follows: If we have a relation R then

(1) $w \in \langle R\rangle \implies uwu^{-1} \in \langle R\rangle$
(2) $w_1, w_2 \in \langle R\rangle \implies w_1w_2 \in \langle R\rangle$
(3) $w \in R \implies w^{-1} \in \langle R\rangle$

THEOREM 4.3.23 (Tietze). *Given G with 2 presentations $G \cong \{X \mid R\}$ and $G \cong \{Y \mid S\}$. These presentations are obtained from each other by a sequence of:*

(1) Add a generator u and a relation $u = e$.
(2) Conversely, if a generator u appears only in the relation $u = e$, delete it.
(3) If u, v are in a set of relations, add uv.
(4) Conversely, if u, v, uv are in a set of relations, delete uv.
(5) If u is in a relation, add gug^{-1} for $g \in G$.
(6) Coversely, if u, gug^{-1} are in a relation, delete gug^{-1}.
(7) If u is in a relation, add u^{-1}.
(8) Conversely, if u, u^{-1} are in a relation, delete u^{-1}.

REMARK 4.3.24. These are all of the noncommutative analogs of Gaussian-Jordan Elimination.

So we can enumerate all possible presentations of a group, but the trouble is that we will never know what we are going to get!

The following is an unsolved problem:

CONJECTURE 4.3.25. *Suppose G is a nontrivial group. Now add to its presentation one generator and one relation. The result is still nontrivial.*

The conjecture is known to be true if G is finite or if G has no elements of finite order. Everything in between these two extremes is unknown.

4.3.2. Combining Groups to get Larger Groups.

DEFINITION 4.3.26. There are two basic ways of combining two groups $G = \{X \mid R\}$ and $H = \{Y \mid S\}$: the *direct product* $G \times H$ and the *free product* $G * H = \{X \cup Y \mid R \cup S\}$.

EXAMPLE 4.3.27. $F_k * F_\ell = F_{k+\ell}$.

EXAMPLE 4.3.28. $\mathbb{Z}_2 * \mathbb{Z}_2 = \{a, b \mid a^2, b^2\}$. This is, in fact, an infinite group! However, $\mathbb{Z}_2 * \mathbb{Z}_2 / \langle (ab)^k \rangle \cong D_{2k} = \{\sigma, \tau \mid \sigma^k = e, \tau^2 = e, \tau\sigma\tau^{-1} = \sigma\}$

EXERCISE 4.3.29. Show that the two presentations in the previous example are isomorphic.

REMARK 4.3.30. If we add more relations to a presentation, we get a quotient group.

EXAMPLE 4.3.31. $G * \{e\} = G$

REMARK 4.3.32. We did not show that $G*H$ is independent of the presentations of G and H. There are two ways to check this: by using Tietze moves, or by using the diagrammatic definition of the free product.

REMARK 4.3.33. From the point of view of generators and relations, the direct product is

$$G \times H = \{X, Y \mid R, S, [u, v] \text{ for } u \in X, v \in Y\}.$$

EXAMPLE 4.3.34. $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{x, y \mid x^2, y^2, xyx^{-1}y^{-1}\}$.

So in fact, $G \times H$ is a quotient of $G * H$.

The diagrammatic definition of the free product $G * H$ is as follows: Given homomorphisms $\alpha : G \to M$ and $\beta : H \to M$, then there exists a unique homomorphism $\rho : G * H \to M$ where

$$\rho(g) = \begin{cases} \alpha(g) & g \in G \\ \beta(g) & g \in H \end{cases}.$$

We can write $\rho = \alpha * \beta$.

$$\begin{tikzcd}
G \arrow[r, hook] \arrow[dr, "\alpha"'] & G * H \arrow[d, "\exists!"] & H \arrow[l, hook'] \arrow[dl, "\beta"] \\
& M &
\end{tikzcd}$$

FIGURE 4.3.35.

If we assume the condition on M that $\alpha(G)$ commutes with $\beta(H)$, then in fact we have the direct product $G \times H$.

The *free product with amalgamation* is given as follows: Given two groups $G = \{X \mid R\}$, $H = \{Y \mid S\}$ and a group $A = \{Z \mid \cdot\}$ equipped with homomorphisms $i : A \to G$, $j : A \to H$ (in many applications, A is a subgroup of both G and H), then

$$G *_A H = \{X, Y \mid R, S, i(u) = j(u) \text{ for } u \in Z\}$$

is the free product with amalgamation.

EXAMPLE 4.3.36. $G *_G G = G$.

EXAMPLE 4.3.37. $G *_{\{e\}} H = G * H$.

4.4. Van Kampen's Theorem

Now we apply all of the previous definition.

THEOREM 4.4.1 (Special Case of Van Kampen's Theorem). *Let $X_{,x} = A_{,x} \cup B_{,x}$ where A, B are open, $x \in A \cap B$, and $A, B, A \cap B$ are path connected. Assume $A \cap B$ is simply connected, then $\pi_1(X, x) = \pi_1(A, x) * \pi_1(B, x)$.*

COROLLARY 4.4.2. $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.

EXAMPLE 4.4.3. $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$. However, $\pi_1(S^1 \vee S^n) = \mathbb{Z}$ for $n > 1$ since S^n is simply connected for $n > 1$.

REMARK 4.4.4. Note that if $\pi_1(A \cap B) \neq 0$, this cannot be the right formula because it counts $\pi_1(A \cap B)$ twice.

THEOREM 4.4.5 (Van Kampen's Theorem). *Let $X_{,x} = A_{,x} \cup B_{,x}$ where A, B are open, with $x \in A \cap B$ and $A, B, A \cap B$ are path connected. Then we have $\pi_1(X) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$ using the inclusions $i_* : \pi_1(A \cap B) \to \pi_1(A)$, $j_* : \pi_1(A \cap B) \to \pi_1(B)$.*

REMARK 4.4.6. i_*, j_* may fail to be injective.

EXAMPLE 4.4.7. Take the torus and puncture it. This is homotopy equivalent to $S^1 \vee S^1$.

FIGURE 4.4.8.

Then the fundamental group of the punctured torus is $\mathbb{Z} * \mathbb{Z}$. To reobtain the Torus, we fill the hole back in: Torus = Punctured Torus $\cup_{S^1} D^2$. Then by Van Kampen's Theorem, $\pi_1(\text{Torus}) = \mathbb{Z} * \mathbb{Z} *_{\mathbb{Z}} \{e\}$. Well, $j_* : \mathbb{Z} \to \{e\}$ is empty, and $i : S^1 \to S^1 \vee S^1$ sends S^1 around $aba^{-1}b^{-1}$, so

$$\pi_1(\text{Torus}) = \{a, b \mid aba^{-1}b^{-1} = e\} = \mathbb{Z} \times \mathbb{Z},$$

as we already knew.

EXERCISE 4.4.9. Compute the fundamental group of the Klein Bottle.

We quickly sketch a proof of Van Kampen's Theorem.

Recall that there are two main things at work: the generators coming from the two sides; and the relations identifying where the generators overlap. We already looked at the first, where we broke up a loop to get paths on the two sides, and modified these to get loops on each side.

Well, we can represent the generators by $I \to X$ and relations by $I \times I \to X$. We can, using compactness, break the square into a bunch of little squares such that each little square maps to A or B. One can easily rearrange these squares so that each maps to A or B and the overlaps to $A \cap B$, where the corner of each little square maps back to the basepoint.

EXAMPLE 4.4.10. A harder example of using Van Kampen's Theorem: Take a surface of genus g Σ_g. We can compute $\pi_1(\Sigma_g)$ as follows: puncture it, then the result is homotopy equivalent to $\bigvee S^1$, so we get a result that is almost F_{2g}:

$$\pi_1(\Sigma_g) = \{x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] = e\}.$$

4.5. An Example from Knot Theory

We present a somewhat difficult example of using Van Kampen's Theorem from Algebraic Knot Theory.

DEFINITION 4.5.1. A *classical knot* is $K \hookrightarrow S^3$ where $K \approx S^1$.

REMARK 4.5.2. We can also view knots as in $\mathbb{R}^3$, but topologists prefer a compact space so we use $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

EXAMPLE 4.5.3. The bare circle in S^3 is called the trivial knot or "unknot".

EXAMPLE 4.5.4. The trefoil:

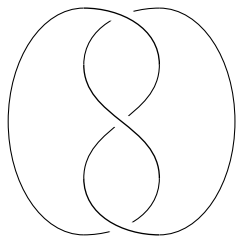

FIGURE 4.5.5.

EXAMPLE 4.5.6. The figure "8"

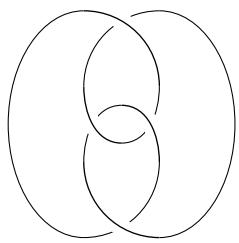

FIGURE 4.5.7.

However, the definition is a bit unsatisfying, because it allows all kinds of wacky objects.

EXAMPLE 4.5.8. Take a "string" and give it one loop, then two loops, then three loops, etc. then by the end loop it an infinite amount of times. Then connect the two ends to bring it back. The result has "infinite knottedness".

We usually want to avoid such things, so there are often various conditions which exclude such objects:

(1) Use only piecewise-linear subsets of S^3 homeomorphic to S^1.
(2) Use differentiability (with $\frac{\partial}{\partial t}$ not zero).
(3) Assume the knot has a thickening to a copy of $S^1 \times D^2$.

We will knot show this, but assuming either piece-wise linearity or differentiability implies that we can have a thickening.

Like with many things, we would like to classify all knots. One way is by number of crossings. Now the trefoil is the only knot with three crossings and the figure "8" is the only knot with four. The number of knots increases wildly with the number of crossings. But this is not a good way to classify knots – after all, we can start with a trillion crossings, but move things around and end up with no crossings! So it is not obvious how many crossings a knot has. Let us make rigorous these "moves".

DEFINITION 4.5.9. The *Reidemeister moves* are as follows:

(1) Take a strand and give it a twist. This gives an additional crossing

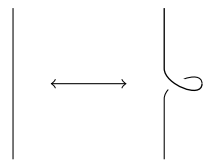

FIGURE 4.5.10.

(2) Take two strands and pull one over the other. This gives two additional crossings.

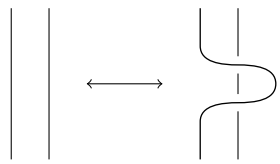

FIGURE 4.5.11.

(3) Take two strands that have a crossing, and a third that crosses the other two below their crossing; move the third so that it crosses the other two above their crossing.

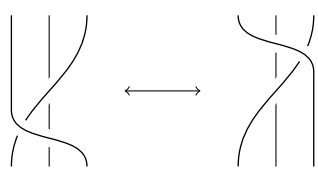

FIGURE 4.5.12.

These produce equivalent knots, so we can use them to show that two knots are the same. One can prove that all moves can be decomposed into combinations of these three moves. So classification of knots is equivalent to classification of projections of knots up to the Reidemeister moves.

Our main goal here is to show that the trefoil is not the same as the unknot. It is not always easy to show that two knots are different. We will use the fundamental group to show this.

First, let us give another description of what it means for two knots to not be the same.

DEFINITION 4.5.13. Given two knots $K_1, K_2 \hookrightarrow S^3$, we will say they are *equivalent* if there is a homeomorphism $\varphi : S^3 \to S^3$ with $\phi(K_1) = \phi(K_2)$.

This is not the sharpest definition we can use, as we aren't bothering to distinguish a knot from its mirror image. Of course, if we prove that the trefoil is not equivalent to the unknot, then we can prove that the mirror image of the trefoil is not equivalent to the unknot, and of course the unknot is equivalent to its own mirror image.

So how will we use the fundamental group? The idea is that we will distinguish knots by distinguishing between their "complements".

DEFINITION 4.5.14. Given a knot $K \hookrightarrow S^3$, its *complement* is $W = S^3 \setminus K$.

Well, this is a complicated space! It's not immediately obvious what this looks like. But we can distinguish the W_1 and W_2 to distinguish K_1 and K_2, since $\phi(K_1) = K_2$ implies that $\phi_1(W_1) = \phi_1(W_2)$.

Now let $G_1 = \pi_1(S^3 \setminus K_1)$ (then we call G_1 the *group of the knot* K_1) and $G_2 = \pi_1(S^3 \setminus K_2)$. Then if $G_1 \ncong G_2$ then K_1 and K_2 are inequivalent knots.

Well, this is turning a problem in geometry into a problem in algebra, and this raises a lot of questions. We can ask, "are there different knots with the same group?" and there are examples of that, but there are no examples of what are called "prime knots" that have the same group even if they are different.

Many people tend to prefer to work with closed sets than with open sets, so what we can do is "thicken" up the knot to a copy of $S^1 \times D^2$, but then take the closed complement of the knot as $\overline{W} = S^3 \setminus \mathrm{Int}(S^1 \times D^2)$, which is of course homotopy equivalent to $W = S^3 \setminus (S^1 \times D^2)$, so they have the same fundamental group.

What we will end up showing is that the fundamental group of the unknot is $\mathbb{Z}$ but the fundamental group of the trefoil is not abelian.

A basic fact is as follows: we can decompose S^3 into two torii glued along their common torus:

PROPOSITION 4.5.15. $S^3 = S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1$.

PROOF. Take a copy of D^2 and revolve it around a vertical axis to get a volume $D^2 \times S^1$.

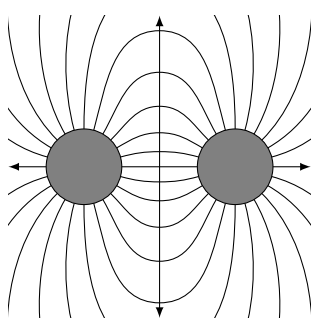

FIGURE 4.5.16.

Next, take an infinite family of curves from the surface of the torus on one side to the surface on the other side; rotating these around gives an infinite family of disks D^2, each of which intersects the axis at one point. However, since the two ends of the vertical axis meet at ∞, it forms a loop S^1, so we get $S^1 \times D^2$; this covers the entire space outside of our first torus $D^2 \times S^1$. So the outside of the torus is another torus. $\square$

REMARK 4.5.17. The order is important, as we can do $S^1 \times D^2 \cup_{S^1 \times S^1} S^1 \times D^2 = S^1 \times (D^2 \cup_{S^1} D^2) = S^1 \times D^2$.

REMARK 4.5.18. $D^4 \approx D^2 \times D^2$, so $S^3 = \partial D^4 \approx \partial(D^2 \times D^2) \approx (\partial D^2) \times D^2 \cup_{(\partial D^2)\times(\partial D^2)} D^2 \times (\partial D^2) \approx S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1$.

To check, let us look at the fundamental group of S^3, which we know to be 0. Take $\pi_1(S^3) = \pi_1(S^1 \times D^2) *_{\pi_1(S^1\times S^1)} \pi_1(D^2 \times S^1) = \mathbb{Z}_1 *_{\mathbb{Z}\times\mathbb{Z}} \mathbb{Z}_2$ where $\mathbb{Z}\times\mathbb{Z} \overset{(0,1)}{\to} \mathbb{Z}_1$ and $\mathbb{Z} \times \mathbb{Z} \overset{(0,1)}{\to} \mathbb{Z}_2$, so we get $\{a, b \mid a = b = e\} = 0$. So our formula is consistent.

We can place the unknot in the middle of $S^1 \times D^2$, so its complement is $D^2 \times S^1$, so the fundamental group is $\mathbb{Z}$.

Now, there are a family of knots called the (p, q) *torus knots*: take a curve with slope p/q with respect to some coordinate system and wrap it around the torus until it meets back up with itself. We will show that the trefoil is equivalent to the $(2, 3)$ torus knot.

Formally, start with the construction of the torus taking $I \times I$ parametrized by a, b so that $(a, 0) \sim (a, 1)$ and $(0, b) \sim (1, b)$. Then we can just start at $(0, 0)$ and draw the line with the slope p/q, jumping to the other side when we hit an edge.

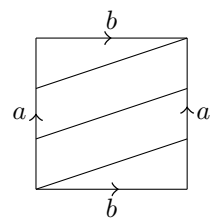

FIGURE 4.5.19.

Now imagine the trefoil, or any generalization thereof with an odd number q of crossings, and run a small tube through the crossings; at the ends, expand the tube out and bring it down to meet so that it becomes a torus. Then the crossings form a $(2, q)$ torus knot.

Hence we can get $S^3 \setminus K_{(2,3)} = (S^1 \times D^2 \setminus K_{(2,3)}) \cup_{S^1\times S^1 \setminus K_{(2,3)}} (D^2 \times S^1 \setminus K_{(2,3)})$. Well, $\pi_1(S^1 \times D^2 \setminus K_{(2,3)}) = \pi_1(D^2 \times S^1 \setminus K_{(2,3)}) = \mathbb{Z}$, but since we have that $S^1 \times S^1 \setminus K_{(2,3)} \approx S^1 \times (0, 1)$, $\pi_1(S^1 \times S^1 \setminus K_{(2,3)}) = \mathbb{Z}$ as well. So we have $\pi_1(S^3 \setminus K_{(2,3)}) = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$. The important part is to see how they are glued. Well, $S^1 \times S^1 \setminus K_{(2,3)}$ goes around a 2 times and around b 3 times, so that we get $\pi_1(S^3 \setminus K_{(2,3)}) = \{x, y \mid x^2 = y^3\}$. In general, $\pi_1(S^3 \setminus K_{(p,q)} = \{x, y \mid x^p = y^q\}$.

We would like to show that $G_{p,q} = \{x, y \mid x^p = y^q\}$ is not abelian. In fact, if we take $G_{p,q}$ and quotient by the commutators, we get $G_{p,q}/[\cdot, \cdot] = \{x, y \mid px - qy\} = \mathbb{Z}$. It is a fundamental result that every knot group abelianizes to the integers.

Let us show the special case for $G_{2,3}$.

PROPOSITION 4.5.20. *$G_{2,3}$ is not abelian.*

PROOF. $G_{2,3} = \{x, y \mid x^2 = y^3\}$ surjects onto $D_6 \cong S_3$, so we have the non-abelian quotient S_3, so $G_{2,3}$ is not abelian. □

We will not develop this, but there is another way to distinguish knots using an invariant called Alexander polynomials. In fact, it is faster to just throw away a lot of the group structure, and it is easy to write computer routines to compute Alexander polynomials.

There is a wide array of working with knots and combinatorial group theory, which we will not cover due to lack of time.

As a final example, to get the generators and relations for any knot: pick a direction on the knot, and then split it into directed segments based on the crossings. For example, on the trefoil, we would split it into three directed segments. Then we get three segments, say x_1, x_2, x_3. Now "knit" counterclockwise around each segment. At a crossing we get the following:, say x_i goes over x_j and x_k, splitting them. Then looping under x_i then x_j, then back up under x_i is the same as looping under x_k. So $x_i x_j x_i^{-1} = x_k$. These are called the "knitting relations at crossings", so the fundamental group is $\pi_1 = \{x_1, \dots \mid \text{knitting relations}\}$. This is nice and pretty but difficult to actually compute, as the number of knitting relations is on the order of pq for a (p, q) torus knot.

If we abelianize this group, we get $x_j = x_k$, so we just get $\mathbb{Z}$.

CHAPTER 5

Covering Spaces

5.1. Covering Spaces

Intuitively, a covering space is like an unwrapping of a space. The effect is that this reduces the "number of loops" as we unloop them – that is, π_1.

EXAMPLE 5.1.1. Take S^1 and $\mathbb{R}$. Then we can wrap $\mathbb{R}$ around S^1 using the mapping $f(x) = e^{2\pi ix}$. We can also imagine a bigger circle wrapped around a smaller circle. Using complex notation, we would have $g_k(z) = z^k$, and this wraps the big circle around the small circle k times.

Covering spaces are local homeomorphisms, but it's more than that; it is not sufficient for a local homeomorhpism to be a covering space. For example, consider an open interval covering part of a circle; it is locally homeomorphic but not a covering space.

DEFINITION 5.1.2. A *trivial covering* of a set U a union of disjoint copies of U mapping to U.

For example, given a small open subset of S^1, and we look at the part of $\mathbb{R}$ that maps to it, we get infinitely many small open subsets of $\mathbb{R}$. So we see that local sections of S^1 are trivially covered. This motivates the following definition.

DEFINITION 5.1.3. A *covering space* of X is a map $f : Y \to X$ such that every part of X has an open subset U so that $f^{-1}(U) \xrightarrow{f|_U} U$ is a trivial cover of U.

In other words, the basespace X can be covered by open subsets which are "trivially covered".

EXAMPLE 5.1.4. For the basespace take the torus $X = S^1 \times S^1$ and the space $Y = \mathbb{R}^2$. Well, we can take the axes and wrap around the circles, using $f(x, y) = (e^{2\pi ix}, e^{2\pi iy})$. Geometrically, we could divide the plane into a lattice, then having chosen a small subset of X, the "upstairs" will just be infinitely many copies of the subset, each sitting inside one square of the plane. But globally, the "upstairs" is a plane.

EXERCISE 5.1.5. Show that if $Y_1 \to X_1$ and $Y_2 \to X_2$ are covering maps, then $Y_1 \times Y_2 \to X_1 \times X_2$ is a covering map.

EXAMPLE 5.1.6. Take $\mathbb{C} \xrightarrow{e^z} \mathbb{C}^*$. Then given a vertical strip in $\mathbb{C}$, then this maps it to a circular strip in $\mathbb{C}^*$. Well, if we take $f^{-1}(\text{piece of } \mathbb{C}^*) \xrightarrow{f|_U} (\text{piece of } \mathbb{C}^*)$. Then, well, the inverse for this mapping, $(f\ |_U)^{-1}$, is the generalization of the natural logarithm.

DEFINITION 5.1.7. The *degree of a cover* f is the size of the inverse set $f^{-1}(p)$.

Exercise 5.1.8. Show that this is independent of the choice of p.

Exercise 5.1.9. Take $Z \xrightarrow{g} Y \xrightarrow{h} X$, where g, h are covering maps. Show that $h \circ g$ is a covering map and that $\deg(h \circ g) = \deg(h)\deg(g)$.

Example 5.1.10. Recall the construction $\mathbb{R}\mathrm{P}^n = S^n/x \sim (-x)$. Then S^n is a covering of $\mathbb{R}\mathrm{P}^n$ using the 2-to-1 quotient map q of degree 2.

Example 5.1.11. The torus covers the Klein bottle as follows: take the torus and map around the Klein bottle twice. This works since going through the flip twice takes you back to where you started. This also gives the conclusion that $\mathbb{R}^2$ is a cover of the Klein bottle.

We will discuss this in a few minutes, but this is the unique simply connected covering of the Klein bottle.

Example 5.1.12. Take a figure 8 and draw one of the loops horizontal and the other vertical. Take the basepoint to be the vertex. Now, take a horizontal line and wrap it around the horizontal loop, then there are infinitely many points at which it hits the basepoint. This is not a covering map, but if we put a vertical loop at every hitting point, then this gives a cover of infinite degree. But we could also have one of these vertical loops unwind to become a vertical line, with horizontal loops at all its hitting points, and so on and so forth. So we quickly obtain uncountably many different covers of the figure 8. If we unwind all the loops we get something like a televesion antenna. But we could also take a circle wrapping around the horizontal loop twice, then at each hitting point add a vertical loop as an "ear". Then this is a covering of degree 2.

The *lifting problem* asks if what we see "upstairs" is the same as what we see "downstairs". Formally, it is as follows: Let $Y_{,y} \xrightarrow{g} X_{,x}$ be a covering map with $g(y) = x$ and $A_{,a} \xrightarrow{f} X_{,x}$ be a continuous map with $f(a) = x$.

$$\begin{tikzcd}
 & Y_{,y} \arrow[d, dash, "g"] \\
A_{,a} \arrow[ur, "\hat{f}?"] \arrow[r, dash, "f"'] & X_{,x}
\end{tikzcd}$$

Figure 5.1.13.

The question is that can we find copies of images of f in Y? We say $\hat{f}$ a lift of f if $g \circ \hat{f} = f$, and lift$_+$ if it preserves the basepoint. But say that the image of A "touches itself" in such a way that in the "upstairs", it is not touching itself but touching its adjacent copies. Then there can be no lift since there is no "upstairs" where the copy of the image "touches" itself. So when does a lift exist, and if it does, is it unique?

THEOREM 5.1.14 (Lifting Theorem). *A necessary and sufficient condition to have a lift$_+$ is that* $\mathrm{Im}(f_*) \subset \mathrm{Im}(g_*)$. *Lift$_+$s, when the exist, are unique.*

PROOF. Turn the lifting diagram into a diagram about their fundamental groups, with $\pi_1(A,a) \xrightarrow{f_*} \pi_1(X,x)$ with $\pi_1(Y,y) \xrightarrow{g_*} \pi_1(X,x)$. Then if there is a loop, we also have $\pi_1(A,a) \xrightarrow{\hat{f}_*} \pi_1(Y,y)$ with $g_* \circ \hat{f}_* = f_*$. This implies that $\mathrm{Im}(f_*) \subset \mathrm{Im}(g_*)$.

We omit the proof of sufficiency and uniqueness. □

COROLLARY 5.1.15. *If A or X are simply connected, then there exists a lift$_+$.*

COROLLARY 5.1.16. *Say we have $Y \xrightarrow{f} X$ as a covering map. Then $\pi_1(Y) \xhookrightarrow{f_*} \pi_1(X)$.*

PROOF. If $[\alpha] \in \mathrm{Ker}(f_*) : \pi_1(Y) \to \pi_1(X)$, then $f \circ \alpha$ is a loop in Y which extends to $D^2 \to X$. This lifts$_+$. By uniqueness, it is filling in the same loop α. □

DEFINITION 5.1.17. A cover $Y \to X$ is called a *universal cover* if $\pi_1(Y) = 0$.

Intuitively, this is what happens when you unwrap everything.

EXAMPLE 5.1.18. $R^1 \to S^1$, $R^2 \to$ Torus, $R^2 \to$ Klein, $S^n \to \mathbb{R}\mathrm{P}^n$, and the full antenna, are all universal covers.

Let $X_{,x}$ be a path-connected space. Then there exists a mapping from covers to subsets of the fundamental group $\{\text{covers}_+ \text{ of } X\} \to \{\text{subgroups of } \pi_1(x,x)\}$ given by $(f : Y \to X) \mapsto (f_*(\pi_1(Y,y)) \subset \pi_1(X,x)$.

THEOREM 5.1.19 (Classification of Covering Spaces). *The map given above is a 1-to-1 and onto correspondence. That is, each unwrapping leaves a unique subgroup of the fundamental group.*

EXAMPLE 5.1.20. We have $\pi_1(S) \cong \mathbb{Z}$. Subgroups of $\mathbb{Z}$ are $k\mathbb{Z}$, $k = 0, 1, 2, \ldots$. Well, $\mathbb{R}$ using the exponential map gives the covering space with fundamental group 0. Then, we have S^1 using the g_k mapping wrapping around k times. Well, the map $(g_k)_* : \mathbb{Z} \to \mathbb{Z}$ is in fact multiplication by k, that is, the image of $(g_k)_* = k\mathbb{Z} \subset \mathbb{Z}$. These are all the covering spaces of the circle up to homeomorphism, for we have classified all the subgroups.

DEFINITION 5.1.21. Given two covers $Y_1 \xrightarrow{g_1} X$ and $Y_2 \xrightarrow{g_2} X$, a *map of covers* h is a continuous function $h : Y_1 \to Y_2$ with $g_2 \circ h = g_1$.

EXAMPLE 5.1.22. Take $\mathbb{R}$ covering S^1 via $e^{2\pi i x}$ and S^1 covering S^1 via $g_5(z) = z^5$. Then $h(x) = e^{2\pi i \frac{x}{5}}$.

DEFINITION 5.1.23. An *equivalence of covers* h is a map of covering spaces which is a homeomorphism.

So the question is, when is there a map of covers between covering spaces of X?

THEOREM 5.1.24. *There exists a map$_+$ of covers between two covering spaces of X if and only if* $\mathrm{Im}(g_1)_* \subset \mathrm{Im}(g_2)_*$.

PROOF. Take the lifting diagram and rotate it, then apply the lifting theorem. □

Corollary 5.1.25. *Covering spaces$_+$ are equivalent if and only if they correspond to the same subgroup of $\pi_1(X)$.*

So covering spaces are completely determined by subgroups of the fundamental group.

The reason the figure 8 has so many covering spaces is because there are a lot of subgroups of the free group on two generators.

5.2. Covering Translations

Given a covering map $Y \xrightarrow{f} X$, what are its symmetries? Well, there can't be too many. Let's do an example.

Example 5.2.1. Start again with $\mathbb{R} \xrightarrow{f(x)=e^{2\pi i x}} S^1$. We need a map $h : \mathbb{R} \to \mathbb{R}$ that is compatible with $f \circ h = f$. So $h(x) = x + k$ for $k \in \mathbb{Z}$. The symmetries are equal to translation by an integer.

Notice that the symmetries is just the fundamental group. In general, we will get a reformulation of the fundamental group as symmetries. This gives an easy way to compute the fundamental group.

Definition 5.2.2. h is a *covering translation* means $h : Y \to Y$ with $f \circ h = f$.

The set of all such h with fixed basepoint forms a group under composition, denoted by C.

Example 5.2.3. Take $g_5 : S^1 \to S^1$. There are five symmetries: $C(g_5) = \mathbb{Z}_5$.

Example 5.2.4. Take the cover of the figure 8 using the horizontal line with loops at the hitting points. Then the symmetries is $\mathbb{Z}$. Now unravel one of the loops. Then anything else we do doesn't matter; the group of symmetries is trivial.

So how many symmetries does a covering space have? Take $f : Y_{,y} \to X_{,x}$, then $\Phi : C(f) \to f^{-1}(x)$ given by $\Phi(h) \mapsto h(g)$.

Proposition 5.2.5. *Φ is injective.*

That is, a covering translation is completely determined by what it does to the basepoint, for any basepoint.

Proof. Tilt the lifting diagram and apply the uniqueness of lifts. □

Corollary 5.2.6. *$|C(f)| \leqslant \deg f$.*

Later we will talk about regular covers. They are nice covers – covers with a lot of symmetry. For these, Φ is 1-to-1. They also correspond to normal subgroups of $\pi_1(X)$.

For regular covers, it turns out that given a covering map $f : Y \to X$, then $C(f) \cong \pi_1(X)/(f_*\pi_1(Y))$.

Corollary 5.2.7. *For the universal cover $\tilde{Y}$, $C(f) \cong \pi_1(X)$.*

The problem is that you don't see the correspondence, which we will do next time, but quickly: take a point x in X, then in the universal cover, the paths between a choice of basepoints to all other points upstairs is an enumeration of all the loops.

5.3. Covering Spaces of Graphs and an Application to Algebra

In much of this course we have gone from Topology to Algebra to solve our problems; now we will use Covering Spaces to solve a problem in Algebra. There are algebraic proofs but the original proof was topological and the algebraic proofs are mostly transcriptions of the topological proof into algebraic language.

THEOREM 5.3.1. *Every subgroup of a free group is itself a free group.*

REMARK 5.3.2. The number of generators of a subgroup is often larger than for the original group.

We will in fact get a formula for the number.

DEFINITION 5.3.3. A *graph* $G = (V, E)$ is a set V of vertices and E a set of (undirected) edges. More precisely, E maps to the set of unordered pairs of vertices (note that we allow both elements of the pair to be the same, and having multiple edges that have the same endpoints).

DEFINITION 5.3.4. A *geometric realization* of a graph $X(G)$ is the space formed by $(V \cup (E \times I))/\{\sim\}$ where $\sim$ identifies for each edges e with endpoints v, w, $e \times 0$ with v and $e \times 1$ with w (or $e \times 0$ with w and $e \times 1$ with v).

DEFINITION 5.3.5. A graph G is *connected* if there is a sequence of edges between any two of its vertices.

It is easy to show the following:

PROPOSITION 5.3.6. *G is connected as a graph if and only if its geometric realization $X(G)$ is (path) connected as a space.*

DEFINITION 5.3.7. A graph is called a connected tree if it has no connected series of different edges from any vertex to itself.

EXAMPLE 5.3.8. A graph with a loop is not a tree.

EXAMPLE 5.3.9. A square is not a tree.

DEFINITION 5.3.10. A subgraph of G is a graph containing some of the vertices and edges of G.

The following (which can easily be shown via induction) is a basic fact:

PROPOSITION 5.3.11. *Every connected graph contains a maximal connected tree as a subgraph. This contains all the vertices in the original graph.*

The maximal connected tree is usually not unique, and there exist formulas for how many maximal trees there exist.

COROLLARY 5.3.12. *Every connected graph is homotopy equivalent to a wedge sum of circles (also called a bouquet of circles).*

PROOF. Since a tree is homotopy equivalent to a single point, shrink the maximal tree to a single point. □

Notice that for a connected tree $|E| = |V| - 1$. Thus for any connected graph G the maximal connected tree has $|V| - 1$ edges. Hence $X(G) \underset{h}{\sim} \bigvee_{|E|-|V|+1} S^1$.

This concludes homotopy of graphs. In higher dimensions for complexes, the discussion is much more complicated.

Now given a free group F_m, we can write $F_m = \pi_1(\bigvee_m S^1)$ by Van Kampen's Theorem. So we can now play around with free groups using a representation as a graph.

Recall that for a space X, there is a 1-to-1 correspondence between connected covers of X and subgroups of $\pi_1(X)$. Now given a subgroup $H \subset F_m$, there is a corresponding covering space $f : Y \to X$ where X is the graph realization of F_m such that $\pi_1(Y) \underset{\cong}{\overset{f_*}{\to}} \mathrm{Im}(f_*) \subset \pi_1(Y)$ where $\mathrm{Im}(f_*) = H$.

As an example, take $F_2 = F(a, b)$, and $H = \mathrm{Ker}(\phi)$ where ϕ sends $F_2 \mapsto \mathbb{Z}_2$ via $a \mapsto$ generator and $b \mapsto e$. H is a subgroup of index 2 in F_2. Well, the index of a subgroup is equal to the degree of the covering map. So drawing a and b as loops from the same basepoint, we must go around a twice to cover it. Then at each of the two hitting points, we need to make a copy of b. As a result, H is homotopy equivalent to $F_3 = F(b, a^2, aba^{-1})$.

So in general, given a space $X = \bigvee_m S^1$ and covering map $Y \xrightarrow{f} X$ of degree d so that $\pi_1(Y) \underset{\cong}{\overset{f_*}{\to}} \mathrm{Im}(f_*) \subset \pi_1(Y)$, we have Y is again a connected graph. Well, if d is the index of H in $\pi_1(Y)$, then $Y = \bigvee_{|E|-|V|+1} S^1 = \bigvee_{d(m-1)+1} S^1$. So we conclude that H is a free group on $d(m-1)+1$ generators.

Example 5.3.13. Let H be the set of words on 26 letters of even length. In other words, $H = F_{26}/\{\text{words of even length}\} = \mathbb{Z}_2$. In other words, H has index 2 in F_{26}. So H is free on 51 generators. Specifically the generators are $H = \{a^2, b^2, \ldots, z^2, ab, ac, \ldots, az\}$.

In the examples we have looked at, the subgroups have been normal. There are interesting examples when the subgroup is not normal. The following is such an example.

Example 5.3.14. Take $F_2 \xrightarrow{\phi} S_3 = D_2$ given by $\{a, b \mid \} \mapsto \{a, b \mid a^2, b^3, abab\}$. Well, consider $\phi^{-1}(\mathbb{Z}_2)$. Well, the index of $\mathbb{Z}_2$ in D_6 is 3, so $H = \phi^{-1}(\mathbb{Z}_2)$ also has index 3. As a covering space, H looks like the following: the downstairs is two loops a, b. Upstairs, a is just a loop with one basepoint. b is a loop with three basepoints. Then since $abab = e$ we have two a's going between the other two basepoints of b. So $H = F_4$.

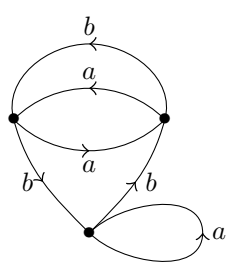

Figure 5.3.15.

This example is actually a counterexample that was used to show the incorrectness of a potential proof to the Poincaré conjecture.

5.4. Regular Covering Spaces

THEOREM 5.4.1. *The following are equivalent for a path-connected covering space $Y \to X$*

(1) $\pi_1(Y) \lhd \pi_1(X)$
(2) For every loop in X, all of its lifts are loops or not loops.
(3) Φ *is surjective (and thus a 1-to-1 correspondence).*

Furthermore, all of these hold if we change the basepoint.

PROOF. (3) $\implies$ (1): Take $y, y' \in f^{-1}(X)$. Then $\pi_1(Y, y') = \gamma^{-1}\pi_1(Y, y)\gamma$ where γ is a path $y \rightsquigarrow y'$. Applying f_* we get $f_*\pi_1(Y, y') = [f(\gamma)]^{-1}\pi_1(Y, y)[f(\gamma)]$. So using different basepoints in Y over Y yields conjugate subgroups. Conversely, every conjugate of $f_*\pi_1(Y, y)$ in $\pi_1(X, x)$ arises this way: For every $[\delta]$ in $\pi_1(X, x)$ lift δ to path $\hat{\delta}$ in Y based at y and ending at a point y'. Then take $\gamma = \hat{\delta}$. Now, Φ surjective means that $Y_{,y}$ is equivalent to a cover $Y_{,y'}$. So they correspond to the same subgroup of $\pi_1(X)$. So these two conjugate subgroups are the same in $\pi_1(X, x)$. That is $f_*(\pi_1(Y, y)) =$ all its conjugates, so it is normal.

(1) $\implies$ (2): Say α is a loop in X. Saying α lifts to a loop at y is to say that $[\alpha] \in \mathrm{Im}(f_*(\pi_1(Y, y)))$, and a similar story holds for α lifts to a loop at y'. If $f_*(\pi_1(Y, y))$ is normal they are the same so α lifts to a loop at one if and only if it lifts to a loop at the other.

(2) $\implies$ (3): We want to find a covering translation moving y to $y' \in f^{-1}(x)$. For that we need to see cover (Y, y) is the same as (Y, y'). So we need to check some π_1. But if $[\alpha]$ lifts to a loop in $Y_{,y}$ that is the same as saying it lifts to a loop in $Y_{,y'}$. So they have the same fundamental group and thus are the same covers.

For the version with moving basepoint, see that (1) must still hold since $\pi_1(X, x) = \omega^{-1}\pi_1(X, x')\omega$. So just apply the lifts of ω to $\pi_1(Y)$. □

DEFINITION 5.4.2. A cover is *regular* if it satisfies any of the above criteria.

THEOREM 5.4.3. $C(f) \cong \pi_1(X)/\pi_1(Y)$.

PROOF. $C(f) \underset{1-1}{\overset{\Phi}{\to}} f^{-1}(x)$ and $\pi_1(X) \overset{\text{lift a loop and take its endpoints}}{\to} f^{-1}(x)$ is 1-1 as well. Need to check that they are compatible with composition. □

It's not always easy to see what a fundamental group or a universal cover is. For example, what exactly is the figure 8 union a disk to fill in $aba^2b^3a^{-1}b^5$?

Unsolved Problem: X is a 2-dimensional complex where $\tilde{X}$ is contractible. Remove a disk to get X'. Must $\tilde{X}'$ be contractible?

EXAMPLE 5.4.4. For $G = \{x_1, \ldots, x_n \mid R_1, \ldots, R_m\}$, X with $\pi_1(X) = G$ is $X = \bigvee_n S^1 \cup (D^2\text{'s corresponding to } R_1, \ldots, R_m)$

EXAMPLE 5.4.5. Take $\mathbb{Z} = \{x \mid \}$, then filling it in with a circle twice is $\{x \mid x^2\}$ so we get $\mathbb{R}\mathrm{P}^2$.

5.5. Construction of Universal Cover (of a path-connected space)

The idea: First, we need to enumerate the points $\widetilde{X}$. At first, we don't mind redundancy. Before we talked about lifting points from X into the covering space.

Well, let $\gamma : I \to X$ be a path from x to some other point z. Then let $y \in f^{-1}(x)$, then there exists a lifting $\hat{\gamma}$ from y to some z' such that $f \circ \hat{\gamma} = \gamma$. So

we can enumerate points in $\widetilde{X}$ as endpoints of paths and we will prescribe paths in $\widetilde{X}$ as lifts of paths in X starting at the basepoint x. This is hugely redundant (since there are many points that go to the same point) but it's a start. So we will use equivalence relations to get rid of the redundancy, and then we have an enumeration of all the points and we can give it a natural topology as being locally homeomorphic to X. It will remain to check that this is a cover of X and that it is simply connected.

This is not crucial, but for simplicity let us suppose that $X_{,x}$ is locally simply connected. Take ${}_iU \subset X$ to be a simply connected open set, with basepoint $u \in {}_iU$. Now consider for each homotopy class of paths pick a path γ from x to u, and for each point $v \in {}_iU$ pick a path δ from u to v.

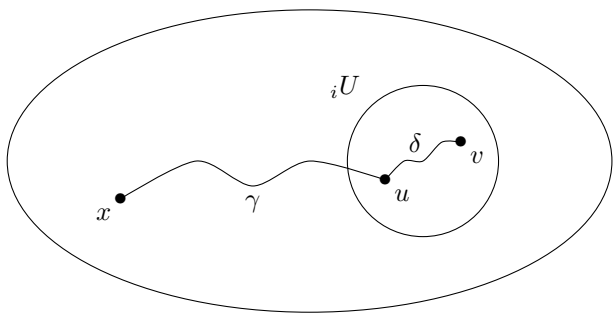

FIGURE 5.5.1.

Set $V_\gamma = \{\gamma \cdot \delta\}$, where γ is chosen and δ is variable (to each point in U). Well there is still some redundancy, say if ${}_iU$ and ${}_jU$ overlap then ${}_i\gamma \cdot {}_i\delta$ and ${}_j\gamma \cdot {}_j\delta$ might lead to the same point. Then we identify them if they are homotopic in X.

Then define $\widetilde{X} = (\bigcup_{i \in I} {}_iV_\gamma)/\text{identification}$.

PROPOSITION 5.5.2. *$\widetilde{X}$ is the universal cover of X.*

We can map $\widetilde{X} \to X$ using the map $f =$ endpoints of the path. Then ${}_iV \xrightarrow{f|} U$ is a one-to-one correspondence (δ is determined by its endpoint in U as U is simply connected). So give ${}_iV$ the same topology as U. It is easy to see that $\widetilde{X}$ is a covering space.

The hard part is showing that the space is simply connected.

Well we showed that the universal cover is characterized among the covers of X by the property that a path β in X that forms a loop, with $\beta \neq e$ in $\pi_1(X)$ has its lift $\hat{\beta}$ to $\widetilde{X}$ satisfying $\hat{\beta}$ is not a loop.

An equivalent formulation is that since β lifts to a loop in Y if and only if $[\beta] \in \mathrm{Im}(\pi_1(Y) \xrightarrow{f_*} \pi_1(X))$ where f_* is injective and $\pi_1(\widetilde{X}) = \{e\}$, then β lifts to a loop only if $[\beta] = e$ in $\pi_1(X)$.

To this end, the lift $\hat{\beta}$ is given as follows: $\hat{\beta}(t) = \beta|_{[0,t]}$ (reparamaterized to go from 0 to 1). We claim that if β is not trivial, that $\hat{\beta}$ is not a loop in $\widetilde{X}$. This is achieved by getting away upstairs from the basepoint.

Another way of saying this is by setting $\widetilde{X}$ as {paths from X to any point of X} quotiented by homotopy of paths with the same endpoints. Then set f to be taking the terminal point. Then if U is a simply connected open subset of X then $f^{-1}(U) = \bigcup_{[\gamma]\in\pi_1(X)} {}_\gamma V$.

5.6. Generalization to Other Covering Spaces

For $H \subset \pi_1(X)$ we want to construct a cover with $\pi_1 = H$. More precisely, we want $Y \xrightarrow{f} X$ with $f_* : \pi_1(Y) \to \pi_1(X)$ injective such that $f_*(\pi_1(Y)) = H \subset \pi_1(X)$.

The trivial case is when $H = \pi_1(X)$, and today we did the universal cover $H = \{e\}$. There are two strategies to construct the rest. The first is to modify the same construction, taking account of H. The second strategy is to write Y as a quotient of $\widetilde{X}$, specifically by some symmetries.

5.6.1. Modifying the previous construction. We set $\widetilde{X}$ to be the set of all paths from x in X quotiented by homotopy of paths with the same endpoints.

Well, we can just set y to be the set of all paths, except with the right quotient. The question is what should be the equivalence relation to quotient by.

Well if we have two paths α, β from x to v that are homotopic we can say $[\alpha\beta^{-1}] \underset{h}{\sim} e$. So the equivalence relation is the one relating for α, β paths with the same endpoints, $[\alpha\beta^{-1}] \in H$. Now the loops in X which lift to loops in Y will be exactly those in H.

5.6.2. As a quotient of $\widetilde{X}$. As an example, we see that we can obtain an intermediate cover $S^1 \xrightarrow{g_k} S^1$. Well, we can write this out by sending $\mathbb{R} \to S^1$ via the map $h_k(x) = e^{2\pi i x/k}$, then $f = g_k \circ h_k$.

We showed that $\pi_1(X)$ acts on $\widetilde{X}$ by covering translations. Then for $H \subset \pi_1(X)$ we can form $Y = \widetilde{X}/H$.

To this end we will review group actions.

Given a group G and a set X, a *G-action* on X is a map $G \times X \xrightarrow{mu} X$ so that $gu = \mu(g, u)$ satisfying $(gh)(u) = g(h(u))$ for $g, h \in G, u \in X$, and $e(u) = u$.

Example 5.6.1. G acts on G by left multiplication.

Example 5.6.2. Given G action on X and $H \subset G$, then H acts on X by restricting the action.

Example 5.6.3. Given homomorphism $H \xrightarrow{\varphi} G$ then if G acts on X then so does H by the induced action $h(u) = \varphi(h)(u)$ for $h \in H, u \in X$.

Remark 5.6.4. This is used to capture the notion of a symmetry of X. Equivalently, $G \xrightarrow{\Phi} \mathrm{Perm}(X)$ which acts on X.

A group action is called *free* if for every $u \in X$, $gu = u \implies g = e$.

If X is a topological space, then an action $G \times X \xrightarrow{\mu} X$ is called *topological* if for each $g \in G$, the map $\mu(g, \cdot) : X \to X$ is continuous. $\mu(g, \cdot)$ is then a homeomorphism with inverse $\mu(g^{-1}, \cdot)$. Alternatively, a group action on a set X is equivalent to a homomorphism $G \to \mathrm{Perm}(X)$, so a topological group action on a space X is equivalent to homomorphism $G \to \mathrm{Homeo}(X)$.

So we are very interested in symmetry, and often symmetry unlocks what is actually going on in the situation. This was one of Einstein's big ideas, the role of symmetry in physics.

EXAMPLE 5.6.5. Take $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (called α and β) acting on S^1, where α is reflection across a vertical axis and β is reflection across the horizontal axis. This action is fixed-point free, that is, there are no fixed points by the entire group. However, this action is not free, since there are elements in the group that fix elements.

EXERCISE 5.6.6. For $G = \mathbb{Z}_p$, p prime, show that fixed-point free is equivalent to free.

There are always subgroups that have fixed-point free but not free actions.

EXAMPLE 5.6.7. Take H to be a proper non-trivial subgroup of G, then look at the cosets $|G : H|$, then G acts on this group by left-multiplication. Then this is fixed-point free but not free.

So lots of things have free actions.

EXAMPLE 5.6.8. $\mathbb{Z}_2$ acting on S^n by the antipodal map. Since there are no fixed points, this is a free action. We can think of this as $\mathbb{Z}_2 = \{\pm 1\}$ under scalar multiplication.

The space $S^n/\mathbb{Z}_2$ is $\mathbb{R}\mathrm{P}^n$.

EXAMPLE 5.6.9. Take $\mathbb{Z}_m$ acting on $S^{2k-1} = \{v \in \mathbb{C}^k \mid \|v\| = 1\}$. Then $\mathbb{Z}_m \cong \{e^{2\pi i k/m} \mid k = 0, \ldots, m-1\}$, the m-th roots of unity. This operates on S^{2k-1} using scalar multiplication.

$S^{2k-1}/\mathbb{Z}_m$ is the *lens space* $L^{2k-1}(m)$, where $L^{2k-1}(m) = \mathbb{R}\mathrm{P}^{2k-1}$.

An example which does not involve abelian groups involves the quaternion numbers $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$. It is well-known that additively $\mathbb{H}$ is the same as $\mathbb{R}^4$ and $\mathbb{C}^2$, but multiplication is given by $i^2 = j^2 = k^2 = -1$, $ij = k, ji = -k$. If $z = a+bi+cj+dk$ then define the conjugate $\overline{z} = a-bi-cj-dk$. Then $z\overline{z} = a^2 + b^2 + c^2 + d^2$. Then $S^3 = \{z \in \mathbb{H} \mid \|z\| = 1\}$ is a (non-commutative) group under multiplication as $\|z\| = 1$.

EXAMPLE 5.6.10. S^3 acts on $\mathbb{H}^k$ by scalar multiplication. Now we can restrict attention to some finite groups in S^3. In particular, consider $\{\pm 1, \pm i, \pm j, \pm k\}$, sometimes called the quaternion 8-group $\mathbb{Q}_8 \subset S^3$. $\mathbb{Q}_8$ acts on the unit sphere in $\mathbb{H}^k$, S^{4k-1}. This is a free action, and we get an interesting space by looking at the quotient, which is also sometimes called $\mathbb{Q}_8$, $S^{4k-1}/\mathbb{Q}_8$. Well we have $\{e\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Q}_8$, so geometrically, we can see a tower of covering spaces out of this, $S^{4k-1} \to S^{4k-1}/\mathbb{Z}_2 \to S^{4k-1}/\mathbb{Z}_4 \to S^{4k-1}/\mathbb{Q}_8$, where each cover is a 2-fold cover for a 8-fold cover overall. The intermediate steps are $S^{4k-1}/\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^{4k-1}$ and $S^{4k-1}/\mathbb{Z}_4 = L^{4k-1}(4)$. The latter is in fact a 3-dimensional space, which is hard to draw. This can be generalized further, to $\mathbb{Q}_{8\times r}$ by introducing more roots of unity.

EXAMPLE 5.6.11. If X is a space, then $\pi_1(X)$ acts as covering translations from $\widetilde{X} \to \widetilde{X}$. For example, for $\widetilde{S^1} = \mathbb{R}$, then using the covering map $f(x) = e^{2\pi i x}$ then the covering trnaslations are $h(x) = x + k$, $k \in \mathbb{Z}$. Then this group is $\mathbb{Z} = \pi_1(S^1)$.

Now we showed that the covering translations don't have any fixed points, so the action of $\pi_1(X)$ on $\widetilde{X}$ is free. Then we can recover $X = \widetilde{X}/\pi_1(X)$. For example, $S^n \to \mathbb{R}\mathrm{P}^2 = S^n/\mathbb{Z}_2$ by the action $\mathbb{Z}_2 = \{e, \text{antipodal map}\}$.

One has to be careful, however, about a technical detail.

EXAMPLE 5.6.12. S^1 acts freely on S^1 but $S^1 \to S^1/S^1 = \{\text{pt}\}$ is not a covering map. We get a similar story for the action $\{e^{2\pi i p/1}\}$. The problem is that the points are "piling up". So to form a good quotient, we need avoid this "piling up".

The condition for not "piling" up for free actions is as follows: a free topological action is called *totally discontinuous* if each $x \in X$ has an open neighborhood U such that the sets $\{gU\}_{g\in G}$ are disjoint. It is easy to see then that covering translations are totally discontinuous, and conversely, given an action of a group G on a space which is free and totally discontinuous, then $q : X \to X/G$ is a covering space.

Observe that if G is finite then total discontinuity is automatic.

A consequence is that if G acts freely and totally discontinuously on a simply connected space X, then for the covering map $X \to X/G$, $X = \widetilde{X/G}$ and $\pi_1(X/G) = G$.

EXAMPLE 5.6.13. $\mathbb{R}/\mathbb{Z} = S^1$, so $\pi_1(S^1) = \mathbb{Z}$.

EXAMPLE 5.6.14. $S^n/\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^n$ so $\pi_1(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_2$.

EXAMPLE 5.6.15. $S^{2k-1}/\mathbb{Z}_m = L^{2k-1}(m)$ so $\pi_1(L^{2k-1}) = \mathbb{Z}_m$.

EXAMPLE 5.6.16. $\pi_1(S^{4k-1}/\mathbb{Q}_8) = \mathbb{Q}_8$.

On an element level, $g \in G$ corresponds to a loop given by the following: in X, look at a path from x to gx, then take the image loop in X/G.

For G finite, there is a one to one correspondence $\{\text{spaces with } \pi_1 \cong G\}$ to $\{\text{simply connected spaces with free } G \text{ action}\}$ via the maps $Y \to \widetilde{Y}$ and $X/G \leftarrow X$.

EXAMPLE 5.6.17. For a lens space,

$$S^{2k-1} \to \mathbb{R}\mathrm{P}^{2k-1} \to L^{2k-1}(4) \to L^{2k-1}(8) \to \dots$$

has no terminal object.

It is likely that the majority of spaces have no symmetry, but we do not have a good way of defining that precisely, and besides, all of the spaces we think about are very pretty and have good symmetry.

Let us return to constructing covers given $\widetilde{X}$ for each $H \subset \pi_1(X)$. Well $\widetilde{X}$ corresponds to $H = \{e\}$. We can construct $X_H = \widetilde{X}/H$, then we have the covering map $X_H \xrightarrow{g} \widetilde{X} = X/G$, with $g_*(\pi_1(X_H)) = H$.

EXAMPLE 5.6.18. Let $H = 2\mathbb{Z} \subset \mathbb{Z}$, then using $\widetilde{S^1} = \mathbb{R}$, then $\mathbb{R}/2\mathbb{Z} = S^1$, but the covering map $g_2(\mathbb{R}/2\mathbb{Z}) \to \mathbb{R}/\mathbb{Z} = S^1$ has $(g_2)_*(\mathbb{Z}) = 2\mathbb{Z} = H \subset \mathbb{Z} = G$.

EXAMPLE 5.6.19. Take $S^{2k-1} \to L^{2k-1}(4)$, $k > 1$, then $\pi_1(L^{2k-1}(4)) = \mathbb{Z}_4$. Then as an intermediate step we have $S^{2k-1}/\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^{2k-1}$ where $\pi_1(\mathbb{R}\mathrm{P}^{2k-1}) = \mathbb{Z}_2 \subset \mathbb{Z}_4$.

All of this is similar to Galois theory. For example, doing Galois theory on $\mathbb{C}[x]$, then the Galois groups are exactly the covering translations if we have the right picture.

5.7. Group Actions on Spaces

DEFINITION 5.7.1. Given two spaces X, Y the *join* of X and Y is $X \star Y = X \times Y \times I/(X \times Y \times \{0\} \sim X \times \{0\}, X \times Y \times \{1\} \sim Y \times \{1\}$.

Obviously $X \subset X \star Y$.

EXAMPLE 5.7.2. $S^0 \star S^0 = S^1$. Similarly $S^0 * S^k = S^{k+1}$. In general, $S^0 \star X = \Sigma X$. So in general inductively $S^i \times S^j = S^{i+j+1}$.

DEFINITION 5.7.3. A *topological group* is a set X equipped with the structures of a topological space and a group such that these structures are compatible in that $\mu : X \times X \to X$ and the inverse $g \mapsto g^{-1}$ are continuous.

EXAMPLE 5.7.4. $G = \mathbb{R}^n$

EXAMPLE 5.7.5. $G = S^1$, the unit complex numbers under multiplication.

EXAMPLE 5.7.6. If G, H are topological groups, so is $G \times H$ in the obvious way. In particular we have the n-torus $\prod_n S^1$.

EXERCISE 5.7.7. Write down the group structure on $R^2 \setminus \{0\} \approx S^1 \times \mathbb{R}$.

These play a big role in physics. The biggest example are matrix groups.

EXAMPLE 5.7.8. The orthogonal, unitary, special orthogonal, special unitary matrix groups O_n, U_n, SO_n, SU_n. All of these are compact. A non-compact space is $GL(n, \mathbb{R})$ which is $\mathbb{R}^{n^2}$.

When we study an action of a topological group G on a space X, we often want this to be compatible with the topology of G. Before when we were given finite G we did not impose a topology on G, but in general if G is a topological group we want $\mu : G \times X \to X$ to be continuous on the product space. In particular in physics we want this for the previously mentioned matrix groups. In fact those are called Lie Groups and have greater structure on top.

We can show that for every group G there is a space with fundamental group G.

The following construction of a space with fundamental group G is due to Milnor.

EXAMPLE 5.7.9 (Milnor). Given a group G with space X with $\pi_1(X) = G$, without details, the idea is give G the discrete topology. Then take the infinite join $Y = \bigcup_{k \in \mathbb{Z}} \bigstar_k G$. We claim that Y is homotopy equivalent to a point; the idea is that Y contains a cone at each stage. For example, taking $G = \mathbb{Z}_2$, then $\bigstar_n \mathbb{Z}_2 = S^n$. Then since S^n sits in S^{n+1} as sort of the equator, then $Y = S^\infty = \bigcup_{k \in \mathbb{Z}} S^k$ is contractible. In particular, $\pi_1(Y) = \{e\}$. Now use left multiplication by G on all coordinates to get a free G-action. Now take the quotient $X = Y/G$, then $\pi_1(X) = G$.

REMARK 5.7.10. This space is called an Eilenberg-Maclane space $K(G, 1)$, which is interesting because it has $\pi_1 = G$ and its universal cover is contractible.

For $G = \mathbb{Z}_2$, this yields $\mathbb{R}\mathrm{P}^\infty = \bigcup_{n \in \mathbb{Z}} \mathbb{R}\mathrm{P}^n$. For $\mathbb{Z}_m$ we get the infinite Lens space. In general it is difficult to see what we will get.

Part III

Manifolds

CHAPTER 6

Differentiable Manifolds and Smooth Functions

6.1. Topological and Differentiable Manifolds

Definition 6.1.1. A *topological manifold* M of dimension n (sometimes written M^n) is a Hausdorff space in which each point has a neighborhood homeomorphic to $\mathbb{R}^n$, or equivalently, an open disk in $\mathbb{R}^n$, or equivalently an open subset in $\mathbb{R}^n$.

This definition is equivalent to saying that the space is covered by copies of $\mathbb{R}^n$, open disks in $\mathbb{R}^n$, or open subsets in $\mathbb{R}^n$.

Example 6.1.2. For every point on a sphere S^n we can project its neighborhood down to a plane on $\mathbb{R}^n$:

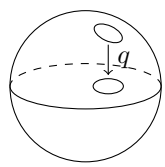

Figure 6.1.3.

Example 6.1.4. The letter X is not a topological manifold, because it is 1-dimensional but at the crossing it does not look like a line.

We need Hausdorff-ness to avoid the following situation:

Example 6.1.5. Take two lines and glue them together except at the point 0. This is not Hausdorff since we cannot separate these two copies of 0. However, it is locally Euclidean, but not Hausdorff!

In order to exclude this wretched space, we need the manifold to be Hausdorff.

We have a *coordinate system* imposed on every neighborhood around the point from the homeomorphism into $\mathbb{R}^n$.

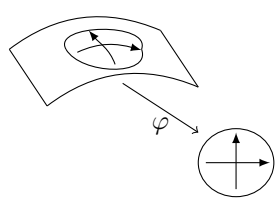

Figure 6.1.6.

On the overlap, we have two coordinate systems. Each is continuous with respect to the other.

For example, for neighborhoods U, V with *coordinate map* φ, ρ then $\rho \circ \varphi^{-1}$ is continuous over $\varphi(U \cap V)$.

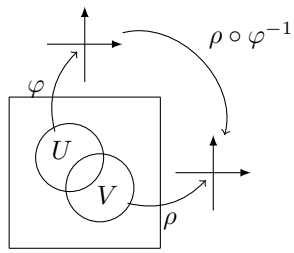

FIGURE 6.1.7.

In general topological manifolds are very hard to work with, so we prefer to work with differentiable manifolds.

DEFINITION 6.1.8. A *differentiable* or *smooth manifold* is a topological manifold such that these functions $\rho \circ \varphi^{-1}$ are smooth over $\varphi(U \cap V)$.

Note that this implies the same for the inverse $\varphi \circ \rho^{-1}$.

Smoothness of these functions is just in the Calculus sense, since we are mapping from Euclidean space to Euclidean space.

We can ask often if manifolds only have one unique differentiable structure. Milnor showed that there were exotic differentiable structures on S^7, and in fact, there are 28. There is a formula for the number of differentiable structures on a sphere, that involves the Bernoulli numbers.

EXAMPLE 6.1.9. Take $S^n \subset \mathbb{R}^{n+1}$, then we can divide it into hemispheres, a northern hemisphere and southern hemisphere, which we can just project down to the hyperplane across the equator. That is, for each $i = 1, \ldots, n+1$ let

$$U_{i+} = \{v \in S^n \mid i - \text{th coordinate is } > 0\}$$

and

$$U_{i-} = \{v \in S^n \mid i - \text{th coordinate is } < 0\}.$$

Then $\varphi_i : (x_1, \ldots, x_{n+1}) \to (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}$. Note that this does not cover the equator, but then we can cut another way to cover them. This uses $2(n+1)$ coordinate "patches". The sphere can be done in two patches but requires a more difficult coordinate system.

We need to show that the coordinate transforms are smooth. Well consider the map $\varphi_{2+} \circ \varphi_{1+}^{-1} : (x_2, \ldots, x_{n+1}) \to (x_1, x_3, x_4, \ldots, x_{n+1})$ where the value of x_1 is $x_1 = \sqrt{1 - (x_2^2 + x_3^2 + \ldots + x_{n+1}^2)}$.

Now this example has been very expensive, as we used many coordinates.

EXAMPLE 6.1.10. In $\mathbb{R}\mathrm{P}^n$, $U_{i+} = U_{i-}$ so we use only $(n+1)$ coordinate patches.

In practice we never show that a manifold is differentiable by checking coordinate maps, as this is too difficult to do. Instead, we invoke theorems that tell us that certain constructions give differentiable manifolds.

But it is very useful to be able to switch between using coordinates and not using coordinates.

A dumb example of a smooth manifold is the tautologous $\mathbb{R}^n$:

EXAMPLE 6.1.11. $\mathbb{R}^n \overset{\varphi=\mathrm{Id}}{\to} \mathbb{R}^n$.

A great deal of mathematics takes place in differentiable manifolds: geometry, analysis, topology.

6.2. Smooth Functions

Once we have a smooth manifold, we can start talking about smooth functions on a smooth manifold. Note that there is no circularity since our smooth manifolds are defined using Advanced Calculus on $\mathbb{R}^n$.

Note that in general we can have different requirements on transform between coordinate systems, such as complex analytic.

DEFINITION 6.2.1. We say $f : M^n \to \mathbb{R}$ is smooth (or differentiable) (here differentiable means infinitely differentiable, i.e. C^∞) if it is smooth as a function of each of the given coordinate systems of M. That is, $f \circ \varphi_\alpha^{-1}$ is smooth for a coordinate map φ_α.

Note that on an overlap $U_\alpha \cap U_\beta$ this is independent of the choice of coordinates φ_α or φ_β, since these are differentiable with respect to each other. That is, we have $f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})$.

An example of a differentiable function that is greatly studied in mathematics, especially in Morse theory is the following:

EXAMPLE 6.2.2. The height function $h : \text{Sphere} \to \mathbb{R}$ or $h : \text{Torus} \to \mathbb{R}$.

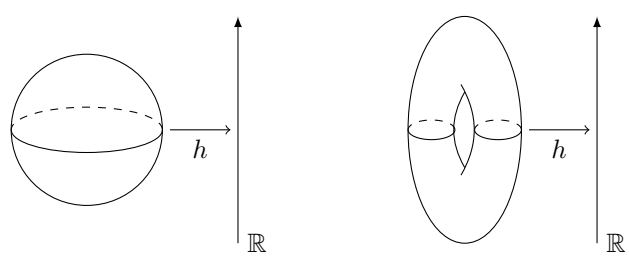

FIGURE 6.2.3.

DEFINITION 6.2.4. $F : M \to \mathbb{R}^k$ where $F = (f_1, \ldots, f_k)$ is said to be smooth (or differentiable) if each f_i is smooth.

More generally, we can discuss what it means for functions from one manifold to another to be smooth:

DEFINITION 6.2.5. $f : M^m \to N^n$ where M^m and N^n are differentiable manifolds, then f is differentiable if it is differentiable when written in terms of coordinate patches of M^m and N^n. That is, for open $U_\alpha \subset M^m$ with coordinate map φ_α and open $V_\beta \subset N^n$ with coordinate map ρ_β then f is differentiable if for $x \in U_\alpha$, $f(x) \in V_\beta$, then near $\varphi_\alpha(x)$ the function $\rho_\beta \circ f \circ \varphi_\alpha^{-1}$ is differentiable.

It is not always crucial to use infinite differentiability but it makes it nice. Given twice differentiability, in fact by adjusting the coordinates we can obtain infinite differentiability.

It is easy to see that composites of smooth functions are smooth. This follows from Advanced Calculus.

6.3. Equivalence of Differentiable Manifolds

In Mathematics we always want to know if things are the same.

Definition 6.3.1. A function $f : M^m \to N^n$ is said to be a *diffeomorphism* if f is a homeomorphism with f and f^{-1} differentiable.

We need to show that f^{-1} is differentiable, because it is possible to have a differentiable function whose inverse is not differentiable:

Example 6.3.2. The function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ given by $f(x) = x^3$ is obviously a homeomorphism and differentiable but $f^{-1} = x^{1/3}$ which is not differentiable at $x = 0$.

In this case there is in fact a diffeomorphism, namely the identity map, but $f(x) = x^3$ does not cut it.

6.4. Excursion: Basic Facts from Analysis

Proposition 6.4.1. *On $\mathbb{R}^n$ there is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $f(v) = 1$ if $\|v\| < \frac{1}{2}$, $0 \leqslant f \leqslant 1$ if $\frac{1}{2} \leqslant \|v\| \leqslant 1$, and $f(v) = 0$ if $\|v\| \geqslant 1$.*

Note that this is not true in complex differentiable functions!

Proof. We first show the case $n = 1$. This implies the case for $n \geqslant 1$, since if we have $f : \mathbb{R} \to \mathbb{R}$ satisfying this property then we can just use $F(v) = f(\|v\|)$.

We start with a function g such that $g = 0$ on $\mathbb{R}^-$ and then rises. So say

$$g(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leqslant 0 \end{cases}.$$

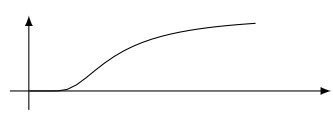

Figure 6.4.2.

Now take the product of this with a shifted mirror image, then we get a sort of smooth bump:

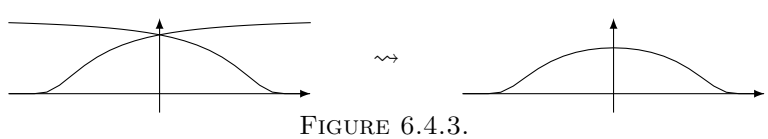

Figure 6.4.3.

Then if we integrate, we get a picture that plateaus after a while; then we multiply the integral by a shifted mirror image to get a mesa. □

Another basic facts are the Inverse Function Theorem, and equivalently the Implicit Function Theorems.

THEOREM 6.4.4 (Inverse Function Theorem). *Suppose we had a function f defined on some neighborhood of 0 in $\mathbb{R}^n$, mapping into some other neighborhood of 0 in $\mathbb{R}^n$ with $f(0) = 0$. The derivative Df is a $n \times n$ Jacobian matrix. If Df is nonsingular at the origin then f has a smooth inverse in a neighborhood of 0.*

In single-variable calculus we can do this on the entire real line, but in greater dimensions we cannot.

EXAMPLE 6.4.5. Take $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^z$. This hits the same point many times as we go around.

However we can talk about how big the open set is, but that is technical and not actually needed.

The idea for the proof is that $f(v)$ is approximated by $(Df(0))(v)$ locally, and this is invertible.

THEOREM 6.4.6 (Implicit Function Theorem). *Suppose we had a function G from $(X = \{\text{neighborhood of 0 in } \mathbb{R}^k\}) \times (Y = \{\text{neighborhood of 0 in } \mathbb{R}^\ell\})$ to $\{\text{neighborhood of 0 in } \mathbb{R}^\ell\}$ with $G(0,0) = 0$. Given that $\left(\frac{\partial G}{\partial Y}\right)_{\ell\times\ell}(0,0)$ is nonsingular, then there exists a smooth function $f : X \to \{\text{neighborhood of 0 in } \mathbb{R}^\ell\}$ so that $G(x, f(x)) = 0$. That is, $G(X, Y) = 0$ defines Y as a smooth function of X.*

So even if it is difficult to write down the description of Y in terms of X this tells you that if we have such a G then there exists such a description.

PROPOSITION 6.4.7. *The Inverse Function Theorem implies the Implicit Function Theorem.*

PROOF. Consider the function $H = (X, G(X, Y))$,

$$\left\{\text{neighborhood of 0 in } \mathbb{R}^{k+\ell}\right\} \xrightarrow{H} \left\{\text{neighborhood of 0 in } \mathbb{R}^{k+\ell}\right\}.$$

Then H is nonsingular at $(0,0)$ since $DH = \begin{pmatrix} I & * \\ 0 & \frac{\partial G}{\partial Y} \end{pmatrix}$. There is an inverse function K, then write $K(X, 0) = (X, f(X))$. This yields $f(x)$ with $G(x, f(x)) = 0$. $\square$

There is another theorem, Sard's Theorem, which we will save for later.

CHAPTER 7

Tangent Spaces and Vector Bundles

7.1. Tangent Spaces

Our next topic is the *tangent space* T_xM, also written $(TM)_x$, of a manifold M at a point $x \in M$.

There are at least three ways of formulating the tangent space:

(1) Think of $M \subset \mathbb{R}^N$, then take the vectors tangent to x, using Advanced Calculus. The problem with this is that it requires us to put our manifolds in Euclidian Space.
(2) Use parametrized (smooth) curves through x. This is difficult because curves do not come with an arithmetic on them, though we can construct one.
(3) Tangent lines used in Advanced Calculus to differentiate along, $\frac{\partial f}{\partial \vec{v}}$. The idea is to define tangent vectors as "rules of differentiation". Putting them together will give the tangent space.

We will use the third approach.

Definition 7.1.1. A *rule of differentiation* at a point $p \in M^n$ is a map

$$\chi : \{\text{smooth functions defined near } p\} \to \mathbb{R}$$

satisfying linearity:

$$\chi(af + bg) = a\chi(f) + b\chi(g), \quad a, b \in \mathbb{R}$$

and the Liebnitz condition:

$$\chi(f \circ g) = f(p)\chi(g) + \chi(f)g(p).$$

In Advanced Calculus we have the following:

Example 7.1.2. $\chi = \frac{\partial}{\partial \vec{v}}$.

Proposition 7.1.3. *The set $(TM)_p = \{\chi \mid \chi$ is a differential rule near $p\}$ is an n-dimensional real vector space.*

Proof. Obviously for rules χ_1, χ_2, $a\chi_1 + b\chi_2$ satisfy linearity and Liebnitz.

Now pick coordinates $\varphi(\cdot) = (x_1, \ldots, x_n)$.

We will show that $\left\{\sum a_i \frac{\partial}{\partial x_i}\right\}_p \in (TM)_p$. There are two claims:

(1) $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ are linearly independent in $(TM)_p$, and
(2) they form a basis for $(TM)_p$.

For (1) just check them on the coordinate functions $x_1, \ldots, x_n$.

For (2) we will show that any χ is given by $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, where $a_i = \chi(x_i)$. Given f a smooth real-valued function defined near p, by Taylor's Theorem f can

be written as

$$f = c + \sum_{i=1}^{n} c_i x_i + \sum x_i x_j g_{ij}(x_1, \dots, x_n).$$

Then

$$\chi(f) = \cancel{\chi(c)} + \sum c_i \chi(x_i) + \sum (\text{functions that are 0 at } p = 0)$$

so

$$\chi(f) = \sum c_i \chi(x_i) = \sum \frac{\partial f}{\partial x_i} \chi(x_i). \qquad \square$$

Let us see how we use these to talk about differentiability at a point.

So given $f : M^m \to N^n$ we want to make sense of $(Df)_p$:

(1) Using coordinates $(x_1, \dots, x_m)$ at p and $(y_1, \dots, y_n)$ at $f(p)$ we can write f as being $f = (f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m))$ we then set $(Df)_p$ equal to the matrix $((\frac{\partial f_i}{\partial x_j}))$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. This is the usual Jacobian matrix in the Advanced Calculus sense. This is nice because it is easily computable, but the disadvantage is that it requires picking coordinates in both M and N.

(2) Intrinsically, without reference to coordinates in M or N, the idea is that a matrix just records a linear map when given a basis in the source and the target. Now we just record Df as a linear map, which, when we introduce coordinates $x_1, \dots, x_m$ in M and $y_1, \dots, y_n$ in N, will give us bases $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ and $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ in terms of which Df is seen as a matrix, the Jacobian matrix.

So we have $p \in M$ and $f(p) \in N$, then we have $(TM)_p$ and $(TN)_{f(p)}$, which we defined intrinsically. Then $(Df)_p : (TM)_p \to (TN)_{f(p)}$ is a linear map defined using the chain rule. We need to describe $((Df)(\chi))(g)$ where g is a real-valued function smooth function near $f(p)$. So we define

$$((Df)(\chi))(g) = \chi(g \circ f).$$

It is easy to check that this satisfies linearity and Liebnitz, so that we get $(Df)(\chi) \in (TN)_{f(p)}$.

The last thing to do is to check that if we give ourselves a basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ for $(TM)_p$ and $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ for $(TN)_{f(p)}$ then the linear map Df with respect to these bases is given by the usual Jacobian matrix.

This is shown by the chain rule.

To discuss Df at more than a point we need to introduce a family of vector spaces TM, the whole *tangent space* of an m-dimensional manifold M. The idea is that there are no good natural way to compare tangent spaces at different points. We need to pick coordinates, but it depends on the choice of coordinates.

So what we will do is form the union of all the tangent spaces $\bigcup_{p \in M} (TM)_p$ and give it a topology as a smooth manifold of dimension $2m$: m dimensions for where it is rooted and m dimensions for the direction.

This is not just a manifold, since it has extra structure, since we can add some of these vectors. So if we use coordinates $x_1, \dots, x_m$ near $p \in M$, for TM use coordinates $(x_1, \dots, x_m, a_1, \dots, a_m)$ where a_i corresponds to $\sum a_i \frac{\partial}{\partial x_j}$ at the point $(x_1, \dots, x_m)$.

So for each point p we get a corresponding neighborhood $U \times \mathbb{R}^m$ in TM. If we take an overlapping set $V \times \mathbb{R}^m$ on the overlap in the underlying manifold there is a change of coordinates given by $(y_1, \dots, y_m) = (f_1(x_1, \dots, x_m), \dots, f_m(x_1, \dots, x_m))$.

In TM the change of coordinates is given by $(f_1, \ldots, f_m, (Df)_{m\times m}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}))$, so we see that this is not a general crossing with $\mathbb{R}^m$; there is additional structure. In Euclidean space this is not the case since we can just take a single vector space based around the origin.

Of course, we can map TM down to M using a natural projection π sending $(x_1, \ldots, x_m, a_1, \ldots, a_m)$ to $(x_1, \ldots, x_m)$ that just forgets the vector spaces.

Notice that here for each $p \in M$, the inverse image $\pi^{-1}(p)$ is an m-dimensional vector space.

Example 7.1.4. In the case of the circle S^1, we can introduce a coordinate of the angle θ. Notice that this is not well-defined, as it is not globally defined, only on pieces of the circle. So we have to be careful. As the angle is increasing, we have a a vector $\frac{\partial}{\partial\theta}$. But even though θ is not well-defined, $\frac{\partial}{\partial\theta}$ is. So in fact $TS^1 = S^1 \times \mathbb{R}$, parametrized by (θ, a) where $\chi = a\frac{\partial}{\partial\theta}$ at point θ.

Remark 7.1.5. In general, the tangent space of the sphere is not equal to the sphere times $\mathbb{R}^n$, except for a few special values: $n = 1, 3, 7$.

7.2. Vector Bundles

This is a key notion that plays a role in many parts of Mathematics.

Before we define it, we already have an example:

Example 7.2.1. If we have $X = M^m$, then $(TM) \xrightarrow{\pi} M$ is a vector bundle.

Example 7.2.2. For any topological space X, take the obvious projection $X \times \mathbb{R}^m \xrightarrow{\pi} X$. This is called the trivial m-dimensional vector bundle.

The fact that the projection $(TS^2) \xrightarrow{\pi} S^2$ is not trivial is of great importance in ODEs.

Definition 7.2.3. A *vector bundle* over a space X is (X, E, π) where X is the base space, E is the total space, $\pi : E \to X$ is a projection map, and $\pi^{-1}(p)$ is an m-dimensional (real) vector space. Now we want these vector spaces to vary continuously for $p \in X$, so we also require local triviality: X can be covered by open sets U such that $\pi^{-1}(U) = U \times \mathbb{R}^n$ in that $\pi^{-1}(p) \cong p \times \mathbb{R}^n$.

The last condition is to exclude the following:

Example 7.2.4. Sitting over $\mathbb{R}$ is a family of vertical lines, then after a certain point they become horizontal lines:

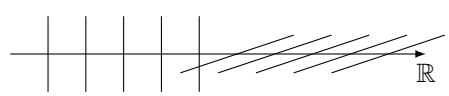

Figure 7.2.5.

So the projection is not continuous.

There is a great deal of study done on invariants on non-trivial vector bundles.

Example 7.2.6. Take an open Mobius strip and project it down to the circle sitting along the middle of the strip. Written in detail, take $S^1 = I/0{\sim}1$, and $E = I \times \mathbb{R}/(0, v){\sim}(1, -v)$. This is a non-trivial 1-dimensional vector bundle.

1-dimensional vector bundles are also called *line bundles*. The only way a line bundle can be non-trivial is that they have Mobius strips built into it.

We will often write $E_p = \pi^{-1}(p)$. These are the fibers, so a vector bundle is a kind of fiber bundle. In general fiber bundles we do not assume a vector space structure.

There is an arithmetic of vector bundles, since many of the things we do for vector spaces can be done for vector bundles, just using reparametrization.

Example 7.2.7. In vector spaces we have the direct sum $V \oplus W$, then if E and F are vector bundles over X then $E \oplus F$ is a vector bundle over X where $(E \oplus F)_p = E_p \oplus F_p$.

Let ε^k be the trivial k-dimensional vector bundle over X, $\varepsilon^k = (X, X \times \mathbb{R}^k, \pi)$.

Example 7.2.8. $TS^n \oplus \varepsilon^1 = \varepsilon^{n+1}$.

A manifold with this property is called stably parallelizable. These manifolds are important but rare.

Example 7.2.9. For $M = S^1 \times \cdots \times S^1$ the m-torus, then $TM = M \times \mathbb{R}^n$ has bases at each point $\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_m}$.

Notice that linear algebra is written into this everywhere.

Example 7.2.10. If X is a point then any vector bundle over X is a vector space.

In general we can think of this as parametrized vector spaces.

If $A \subset X$ and E is a vector bundle over X, we can form $E|_A = (\pi(A), A, \pi_A)$.

Example 7.2.11. Take $S^n \subset \mathbb{R}^{n+1}$. The trivial bundle of this is the one where we take the radial vector pointing outwards. So ε^1 is the space of radial (normal) vectors on the sphere. Then $TS^n \oplus \varepsilon^1 = T\mathbb{R}^{n+1}|_{S^n}$. But then since $T\mathbb{R}^k = \varepsilon^k \times \varepsilon^k = \mathbb{R}^k \times \mathbb{R}^k$, $T(\mathbb{R}^{n+1})|_{S^n} = \varepsilon^{n+1}$.

This works as long as we embed M^n in one dimension higher $\mathbb{R}^{n+1}$ since we can use the outward pointing vector; in two dimensions higher we can't do this.

7.3. Derivatives over Manifolds

The nice thing about vector bundles is that it gives us a language for talking about derivatives over manifolds.

So given a map $f : X \to Y$ of spaces, with E a vector bundle over X and F a vector bundle over Y, a map of vector bundles (over f) is a map $G : E \to F$ compatible with the projections π_E and π_F in that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "G"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Figure 7.3.1.

and G is linear on each fiber. That is, $\pi_F \circ G = f \circ \pi_E$, and $E_p \xrightarrow{G|} F_{f(p)}$ is a linear map.

The main example is the derivative of a function f over manifolds: If f is a smooth manifold $f : M^m \to N^n$ then we have $Df : TM \to TN$, where Df at any point p is just $(Df)_p$. In coordinates, this is the usual Jacobian matrix.

Definition 7.3.2. Given a smooth function $f : M^m \to N^n$ we will say that $p \in M$ is a *critical point* for f if $\mathrm{Rank}(Df)_p < n$.

Remark 7.3.3. If $m = n$ then this is the same as saying that for Df as a matrix that $\det(Df) = 0$. For general m we can reinterpret this as the determinant of minors of Df.

Definition 7.3.4. The points of the form $f(\text{critical point}) \in N$ are called the *critical values*.

Definition 7.3.5. The *regular values* are $N \setminus \{\text{critical values}\}$.

A warning: this includes the points not in $\mathrm{Im}(f)$. Furthermore if $m < n$, $\mathrm{Im}(f) =$ are critical values, so the regular values are $N \setminus \mathrm{Im}(f)$.

We will see for f where $m > n$ that almost all points are regular values, and the behavior there is very nice, in that $f^{-1}(p)$ is a manifold of dimension $m - n$.

Example 7.3.6. For the height function $S^n \xrightarrow{h} \mathbb{R}^1$ then the critical values are the north and south pole. Everywhere else the inverse image is S^{n-1}.

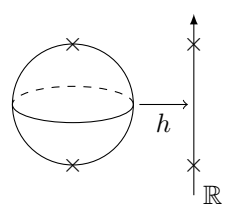

Figure 7.3.7.

Example 7.3.8. For the height function from the vertically-aligned torus to $\mathbb{R}$ there are four critical values.

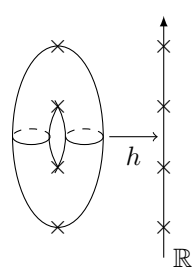

Figure 7.3.9.

The inverse image varies, for at certain points the inverse image is one circle; at other points the inverse image is two circles. At the critical points we have bad behavior: we have points and figure 8s.

EXAMPLE 7.3.10. For $\mathbb{R}^n \xrightarrow{f} 0 \in \mathbb{R}^n$, the critical points are $\mathbb{R}^n$ but the critical values are just $\{0\}$ and the regular values are $\mathbb{R}^n \setminus \{0\}$.

We will need two theorems from Analysis. The first we will not prove, but a good reference for this is Sternberg's book.

THEOREM 7.3.11 (Sard). *Given open $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ and smooth $U \xrightarrow{f} V$ then the set of critical values has measure 0.*

For compact manifolds, and $f : M \to N$, it is easy to see that the critical values is a closed, measure 0 set, so the regular values is an open dense set.

For U a neighborhood of 0 in $\mathbb{R}^m$, and V a neighborhood of 0 in $\mathbb{R}^n$, then for $f : U \to V$ and $f(0) = 0$. Assume $(Df)_0$ has rank n. We want to see that $f^{-1}(0)$ near 0 is a submanifold of dimension $m - n$.

Well if we have coordinates $(x_1, \ldots, x_m)$ in the source and $(y_1, \ldots, y_n)$ in the target, then for the matrix Df without loss of generality suppose the first $n \times n$ block G is non-singular. Consider the new source coordinates $(y_1, \ldots, y_n, x_{n+1}, \ldots, x_m)$. Then

$$\frac{\partial}{\partial(x_1, \ldots, x_m)} = \begin{pmatrix} DG & * \\ 0 & \mathrm{Id}_{m-n} \end{pmatrix}$$

is non-singular. So using these coordinates, f is locally given by projection to the first n coordinates, and $f^{-1}(0)$ is locally just $0 \times \mathbb{R}^{m-n}$.

EXAMPLE 7.3.12. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x_1, \ldots, x_n) = \sum x_i^2$. Then $Df = (2x_1, 2x_2, \ldots, 2x_n)$ which is nonzero except at $(0, \ldots, 0)$. Hence it has maximum rank, and the only critical point is $(0, \ldots, 0)$ with critical value $f(0, \ldots, 0) = 0$. So $f^{-1}(1)$ is a manifold of dimension $(n - 1)$ and is exactly the sphere S^{n-1}.

This gives us a cheap way to see that S^{n-1} is a manifold, without the use of coordinate maps. Very few manifolds come with coordinate maps.

Note that $f^{-1}(0) = \{0\}$, which is not a manifold, so the inverse behaves badly on critical values.

7.4. Manifolds with Boundary

An example of a manifold bounded by a sphere is D^n with boundary S^{n-1}. Another example is a torus with a portion sliced off.

This is similar to the idea of a closed set.

We have the notion of interior points, coordinatized as before. But points on the boundary are coordinatized corresponding to the edge $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_n \geqslant 0$, that is to a point with $x_n = 0$.

Here we call a function $f : A \to \mathbb{R}^k$ for $A \subset \mathbb{R}^n$ smooth if it is the restriction of a smooth function on some open neighborhood of A.

The points on the boundary ∂M of M are coordinatized by $\mathbb{R}^{n-1}$, so form a manifold of dimension $(n - 1)$.

The way we have set this up, the boundary itself has no boundary points. Note that $\partial(\partial M) = \emptyset$.

M is called a *closed manifold* if it is compact and has no boundary, that is $\partial M = \emptyset$.

We saw before that for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ then p is a regular value means that $f^{-1}(p)$ is a manifold of dimension $(n - 1)$. Then $f^{-1}(p) = \delta(f^{-1}(-\infty, p))$ is a manifold with boundary of dimension n.

There is an interesting question of which manifolds are boundaries: given a manifold, is it a boundary of a manifold?

For example, a point is not the boundary of a compact manifold, by elementary arguments.

In 2 dimensions, anything orientable is a boundary, since we can fill it in. It turns out that $\mathbb{R}\mathrm{P}^2$ is not a boundary.

In higher dimensions there is a huge subject studying this known as Bordism and Cobordism Theory.

CHAPTER 8

Degree of Maps

8.1. Degree of Maps (modulo 2)

Now suppose we have smooth $M^n \xrightarrow{f} N^n$ over compact spaces of the same dimension, with N connected. Let p be a regular value of f. Then $f^{-1}(p)$ is a compact 0-dimensional manifold, that is "finitely many points".

Definition 8.1.1. $\deg_p(f) = |f^{-1}(p)| \pmod 2$.

The goals are to show that

(1) The degree mod 2 is independent of the choice of the regular value p, so which justifies defining $\deg(f) = \deg_p(f) \pmod 2$.
(2) If $f \underset{h}{\sim} g$ then $\deg_p(f) = \deg_p(g)$.

For a manifold with boundary W^{n+1} with $M^n = \partial W$ we have TW defined as before using $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ as a basis. Notice that if we restrict to vectors on the boundary, $TW|_{\partial W = M}$ is a vector bundle of dimension $(n+1)$ on M, whereas TM is a vector bundle of dimension n. Now $TW|_{\partial W} = TM \oplus \mathbb{R}$, either an outward pointing vector or an inward pointing vector. The usual convention in mathematics is to use an outward pointing vector.

A vector $\chi = \frac{\partial}{\partial x_{n+1}}$ is outward pointing if it has the property that there is a function f defined near the boundary point with $f \leqslant 0$ but $\chi(f) > 0$.

Example 8.1.2. The tangent space of the unit interval I is $TI = I \times \mathbb{R}$, and $T1 = (1) \times 0$.

We now prove that homotopic maps have the same degree. This is proved by a very simple elementary picture.

Suppose we have a cylinder $M \times I$, and $f : M \times 0 \to N$ and $g : M \times 1 \to N$ and homotopy between them H, where N has the same dimension as M. This picture has dimension $(n+1)$.

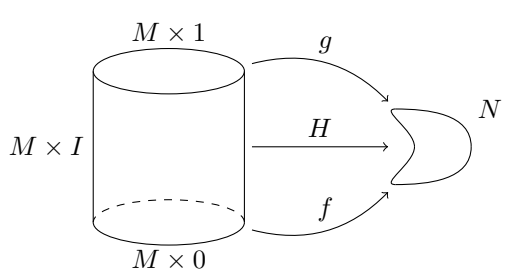

Figure 8.1.3.

So $f = H|_{M\times 0}$ and $g = H|_{M\times 1}$.

Now say f, g, H are smooth, and p a regular value of f, g, H. Now $f^{-1}(p)$ and $g^{-1}(p)$ are each a set of points, but what is $H^{-1}(p)$? It is a manifold with boundary of dimension $(n+1) - n = 1$. Well $\partial H^{-1}(p) = f^{-1}(p) \cup g^{-1}(p)$.

Now the only 1-dimensinal smooth compact manifolds are copies of I or S^1. So we can have lines hanging off the bottom, lines hanving off the top, lines from the top to the bottom, or circles floating in the middle.

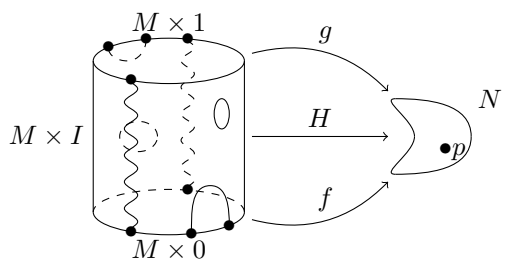

FIGURE 8.1.4.

Now we conclude that the boundary of a 1-manifold is an even number of points, so that $|f^{-1}(p)| = |g^{-1}(p)| \pmod 2$.

Now we did not really use the homotopy H. So more generally, suppose we are given smooth $f : M^n \to N^n$ closed smooth manifolds, with $M = \partial W^{n+1}$, and smooth $F : W \to N$ such that $F|_M = f$. Suppose p is a regular value of f, F. Then $\deg_p(f) = 0 \pmod 2$.

Looking at $F^{-1}(p)$ we have either circles in the middle or arcs hanging off of the boundary. So the number of points on the boundary $\partial F^{-1}(p) = f^{-1}(p)$ is even, since the boundary of a 1-dimensional manifold is even.

This includes the case that $W = M \times I$, where $\partial W = M \times 1 \cup M \times 0$. Then in the case of the cylinder we have $\deg f + \deg g = 0 \pmod 2$ so they are equal modulo 2.

To prove this we needed some strong restrictions. So now we will weaken them.

We assumed that p is a regular value of f on ∂W. We want to justify the assumption that p is a regular value of F on W. We cannot just say to ignore it as we used it in the picture. So we need something clever:

Observe that at a regular value p with f smooth, we can find a neighborhood nearby with coordinates such that any q near p will also be a regular value with $\deg_p(f) = \deg_p(q)$. So degree is locally constant.

Suppose p is a regular value for f, we can find nearby a point g for which g is a regular value of f and for F. Now $\deg_p(f) = \deg_q(f) = 0 \pmod 2$ so we can take a point on the boundary and replace it with a point on the interior.

Before we prove that the degree is independent of the choice of p, we need to discuss uniformity of manifolds.

Intuitively, it means that any point is just as good as any other point.

PROPOSITION 8.1.5. *Let M^n be a connected manifold, and $p, q \in M$. There is a diffeomorphism $\Phi : M \to M$ with $\Phi(p) = q$.*

In the complex world the analog is false: it is not true that on a Riemann surface that we can have a complex diffeomorphism that throws one point to another point.

PROOF. Take D^n and $\rho : D^n \to D^n$ so that $\rho(0, \ldots, 0) = (t_1, 0, \ldots, 0)$ but we want $\rho = \mathrm{Id}$ outside a disk of radius $\frac{1}{2}$. So it will be a push to the right that is dampened as we go out. This is a standard thing to do in ODEs as a flow over vector fields. This tells us that we can move a point to a nearby point.

Define an equivalence relation among points of M by $p \sim q$ if there is such a Φ. Now we can do this on a little disk, so this works.

Notice that the set of points equivalent to p is an open set. So M is decomposed into disjoint open sets, one of which is all of M since M is connected. □

This proof fails in the complex analytic world because ρ would need to be identity everywhere.

We actually need a slight strengthening: We want Φ to be smoothly homotopic to the identity. But this is okay because on the disks we can just flow from time $t = 0$ to time $t = 1$.

So now to show that the degree is independent of p instead of varying the point we vary the function.

Now using uniformity of N, there is a diffeomorphism $\Phi : N \to N$ sending $\Phi(q) = p$, with Φ smoothly homotopic to Id_N.

Consider the function $g = \Phi \circ f$. Notice that f, g are smoothly homotopic since Φ is smoothly homotopic to the identity. Now p is a regular value of g by the chain rule, and from what we already proved we know that $\deg_p(g) = \deg_p(f)$. Well $g^{-1}(p) = (\Phi \circ f)^{-1}(p) = f^{-1}(\Phi^{-1}(p)) = f^{-1}(g)$. So $\deg_p(g) = \deg_q(f)$, so that $\deg_p(f) = \deg_q(f) \pmod 2$.

Now one difficulty is that this was only for smooth maps and smooth homotopies. Another difficulty is that we only did this modulo 2. To get an integer we need to introduce signs and orientations.

But let us talk a little bit about how to drop the smoothness assumption.

One can show in general that for M, N smooth manifolds, any map $f : M \to N$ is homotopic to a smooth map. A similar result can be shown for homotopies of smooth maps.

The idea is that we approximate by a smooth function and then maps that approximates each other are homotopic.

Let us see how to do this for the sphere.

PROPOSITION 8.1.6. *Given a function $f : S^n \to S^m$*

(1) f can be approximated by a smooth function
(2) functions which approximate each other are homotopic.

So any map is homotopic to a smooth map.

We use the Stone-Weierstrass Theorem.

THEOREM 8.1.7 (Stone-Weierstrass). *A function $f : D^n \to \mathbb{R}$ can be approximated by a polynomial.*

So if we have a function $g : S^n \to S^m$, we can extend radially to get a function $G : D^{n+1} \to D^{m+1}$. Then we can approximate each coordinate by polynomials.

So we have a function $S^n \hookrightarrow D^{n+1} \overset{\text{polynomial}}{\to} D^{m+1}$, then look at the composition $S^n \overset{L}{\to} D^{m+1}$, and the smooth function $K(v) = \frac{L(v)}{\|L(v)\|} \in S^m$, which is still an approximation.

Now see that if $f, g : S^n \to S^m \subset \mathbb{R}^{m+1}$ and f and g are close, then we can linearly interpolate between them, that is, look at $H(v, t) = tf(v) + (1 - t)g(v)$.

Now we can make the complaint that this falls off of the sphere, so again we can divide by the norm to get $H(v,t) = \frac{tf(v)+(1-t)g(v)}{\|tf(v)+(1-t)g(v)\|}$.

We can make the same argument for homotopies.

So the swindle is that we can define for any continuous $f : S^n \to S^n$ the degree $\deg(f)$ by replacing f with a homotopic smooth map for which we have already defined the degree. We know that this is well-defined from everything we have said.

So the degree is defined for continuous map, but we can only compute it for smooth maps.

Notice the following:

PROPOSITION 8.1.8. *For* $\deg(f) \neq 0$*, then* f *is surjective.*

PROOF. Suppose $f : S^n \to S^n$ is not surjective, then it is missing a point, so we have $f : S^n \to S^n \setminus \{p\} = \mathbb{R}^n \underset{h}{\sim} \{\text{point}\}$, so $f \underset{h}{\sim}$ constant which has degree 0. □

EXAMPLE 8.1.9. The degree of $S^1 \xrightarrow{f} S^1$, $f(z) = z^k$ is $\deg(f) = k \pmod 2$. Next time we will show that it is k as an integer.

EXAMPLE 8.1.10. Take $S^2 = \mathbb{C} \cup \{\infty\}$, and $f(z) = z^k$ and $f(\infty) = \infty$ then $\deg(f) = k$. The regular values are $S^2 \setminus \{0, \infty\}$.

EXAMPLE 8.1.11. For $S^m \xrightarrow{f} S^m$, $\deg(f) = k$ we can get $S^{m+1} \xrightarrow{F} S^{m+1}$ with $\deg(F) = k$ by taking the suspension of the map and taking f at every level. So for spheres we have maps of any degree we want.

8.2. Orientation

We will first define orientation for vector spaces, then we will talk about vector bundles and then manifolds.

8.2.1. Orientation on Vector Spaces. First let us do it very carefully in the setting of vector spaces. Let V be a finite-dimensional vector space over $\mathbb{R}$. We will define what we mean by an orientation over a vector space first. There are a few ways to do it.

One simple way to do it is follows: An *orientation* of V is an equivalence class of ordered bases of V. So given bases $E = (e_1, \ldots, e_n)$ and $F = (f_1, \ldots, f_n)$ for V, we say $E \sim F$ if the matrix of change of basis has positive determinant. Thus for $\dim V > 0$, there are two equivalence classes of orientations of V. This can be justified by doing a lot of geometry.

REMARK 8.2.1. The set of bases of V is in a one-to-one correspondence with $GL(n, \mathbb{R})$, which has sitting in it the orthogonal matrices O_n. Well the determinant takes $GL(n, \mathbb{R})$ to $\mathbb{R}^\times$, and O_n to $\{\pm 1\} \subset \mathbb{R}^\times$. Notice that both O_n and $GL(n, \mathbb{R})$ both have two components.

So bases of the same orientation class can be varied to each other through bases in that orientation class.

If A, B are oriented, finite dimensional vector spaces, then so is $A \oplus B$. We do the obvious thing, take the basis for A and add the basis for B and check that that is well-defined. A warning: the orientation for $A \oplus B$ may not be the same as that of $B \oplus A$.

EXERCISE 8.2.2. What is the relation?

Given oriented vector spaces E, F, a linear isomorphism $A : E \to F$ is said to be *orientation-preserving* (or reversing) depending on the sign of the determinant of A. We write

$$\varepsilon(A) = \begin{cases} +1 & \det A > 0 \\ -1 & \det A < 0 \end{cases}$$

with respect to given orientations of E, F.

EXAMPLE 8.2.3. $\mathbb{R}^1$ has two orientations. The usual orientation is given by the basis $\{1\}$.

Usually we pick the usual orientation to be pointing to the right.

Notice that an orientation of a vector space V determines one for $V \oplus \mathbb{R}^1$. So if we get an orientation for V and pick the usual orientation for $\mathbb{R}^1$ we get an orientation for $V \oplus \mathbb{R}^1$. If we keep repeating this we get similarly $V \oplus \mathbb{R}^n$.

There is a one-to-one map between orientations for V and orientations for $V \oplus \mathbb{R}^1$, for $V \neq \{0\}$. $\{0\}$ has only one orientation, namely, the one with no basis. This is not good because we want to uniformize.

Some motivation, jumping ahead, is the Fundamental Theorem of Calculus, which says that $\int_0^1 df = f(1) - f(0)$. Basically, we want to be able to give points orientation.

DEFINITION 8.2.4. A *stable orientation* of V is orientation for $V \oplus \mathbb{R}^1$, or an orientation for $V \oplus \mathbb{R}^n$.

A stable orientation is equivalent to an orientation when $V \neq \{0\}$. But every finite dimensional vector space over $\mathbb{R}$ has two stable orientations.

Given v sparse in A, B, C with $A \subset B + C$ where $C = \frac{B}{A}$, then (stable) orientations for any any two of thse determines an orientation for the third. We can show that $B \cong A \oplus C$.

Complex vector spaces come with natural orientation, unlike real vector spaces. If W is a finite dimensional complex vector space, regarding it as a real vector space , it has a natural orientation. W has a complex basis $e_1, \ldots, e_n$. We use the real basis $e_1, \ldots, e_n, ie_1, \ldots, ie_n$.

PROPOSITION 8.2.5. *If we use a different complex basis, we get the same real orientation.*

PROOF. In $\mathbb{R}$ the bases are represented by $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ so the determinant is $a^2 + b^2 > 0$. □

8.2.2. Orientation on Vector Bundles. Given a vector bundle E over a space X, a *orientation* of E is a choice of orientations for each E_p, $p \in X$ which are "locally compatible". So for $E|_U = U \times \mathbb{R}^n$, $p \in U$, $E_p \cong p \times \mathbb{R}^n$ preserving orientation for a fixed orientation of $\mathbb{R}^n$. That is, the orientations agree on overlaps.

If X is a connected space, E has either no or two orientations. If it has one, we can pick the opposite one, at each point. And if it is connected, it is easy to see that the orientation is determined by the orientation for any $E_p, p \in X$. As a proof, look at the subsets of X where the orientations agree, call that U, and where they disagree, call that V. Then U and V are disjoint, and $X = U \cup V$. So one of these is all of X.

Again if E, F are oriented, so is $E \oplus F$, and we work with stable orientations $E \oplus \varepsilon^1$ (or $E \oplus \varepsilon^k$ for $k \geqslant 1$).

A vector bundle can be described by "transition data". Suppose X is covered by open sets U_i. Now for the different U_i's on the overlap, we have a function $U_i \cap U_j \xrightarrow{g_{ij}} GL(n, \mathbb{R})$ where the matrix tells us how $U_i \times \mathbb{R}^n$ identifies to $U_j \times \mathbb{R}^n$, subject to on $U_i \cap U_j \cap U_k$, $g_{ij} g_{ik} = g_{ik}$.

Example 8.2.6. Think of the Mobius strip as a bundle over a cut off circle U and another cut off circle V where the intersection is a piece on the top and a piece on the bottom.

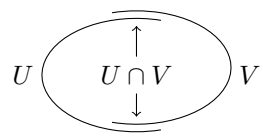

Figure 8.2.7.

Then the way we glue these together is given by $g = +1$ on one component and $g = -1$ on the other, which will give us the twist as we go around.

Actually this bundle has no orientation, since if we give an orientation, when we come back around the orientation will be going the other way.

The condition for orientation is that $\det(g_{ij}) > 0$.

8.2.3. Orientation on Manifolds. For smooth manifolds, the orientations will be given by the coordinates. Before, we had on the overlaps the Jacobian of the change of coordinates, with the condition $D(\varphi_j \circ \varphi_i^{-1}) \neq 0$. Now we just have $\det(D(\varphi_j \circ \varphi_i^{-1})) > 0$.

Giving an orientation for M^n is the same as given an orientation for TM: both are given in terms of the determinant of the derivatives of $\varphi_j \circ \varphi_i^{-1}$. So the Jacobian matrix of change of coordinates is the same as the transition data on TM.

If M is an oriented manifold, we write $-M$ for M with the opposite orientation. This is well-defined even for non-connected manifolds: on each piece we flip the orientation.

Remark 8.2.8. For M 0-dimensional we use stable orientations of $TM = \{0\}$.

So from here on if a manifold is 0-dimension we will take its stabilization.

Proposition 8.2.9. *An orientation for M determines one for ∂M.*

Proof. $T(\partial M) \oplus \varepsilon^1 = TM|_{\partial M}$ using ε^1 to be an outward pointing vector. □

Now the orientation may not be what you think.

Example 8.2.10. Pick an orientation for the interval I, say $TI = I \times \mathbb{R}^1$ using a rightward-pointing vector.

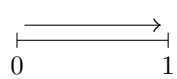

Figure 8.2.11.

Now what about the boundary of the interval? ∂I is $\{1\}$ on one end with the positive orientation, but on the other end if we take the outward pointing vector, that does not agree with the positive orientation. So we get $\{0\}$ with the opposite manifold. So $\partial I = \{1\} \cup -\{0\}$.

So if we take $M \times I$ with orientation going up from 0 to 1 on the interval, then $\partial(M \times I) = M \cup -M$.

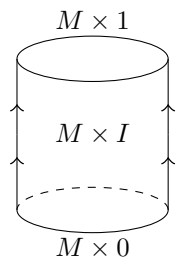

FIGURE 8.2.12.

REMARK 8.2.13. If M is an oriented complex manifold, M as a real manifold has a natural orientation.

8.3. Degree of Maps for Oriented Manifolds

Say M and N are oriented manifolds, and $f : M \to N$ a smooth map. Let $q \in N$ be a regular point; we saw that $f^{-1}(q)$ is a smooth submanifold of dimension $m - n$. We will see that $f^{-1}(q)$ is also oriented. Why is that? What did we do with coordinates? The way we saw that this was a smooth manifold was that for $p \in f^{-1}(q)$ we broke up $(TM)_p$ and we had the coordinates that we broke up into coordinates for $(T(f^{-1}(q)))_p \hookrightarrow (TM)_p$, then we took $(TM)_p/T(f^{-1}(q))_p \xrightarrow{Df} (TN)_q$ as an isomorphism. So we can do the same thing except preserving orientation.

In particular if $m = n$ then $f^{-1}(q)$ is an oriented 0-dimensional manifold and each point and each point $p \in f^{-1}(q)$ gets a sign $\varepsilon(p) = \pm 1$ depending on its orientation.

We need the following:

PROPOSITION 8.3.1. *An oriented compact 1-dimensional manifold is a union of intervals and circles.*

COROLLARY 8.3.2. *The boundary of an oriented compact 1-dimensional manifold has a total of 0 points when these are counted with signs.*

For example ∂I will be one end point minus the other. The total sign is 0.

So now for oriented manifolds, we redo our theory of degree. We now prove the following:

DEFINITION 8.3.3. For oriented closed smooth manifolds M^n, N^n and $M \xrightarrow{f} N$ a smooth map, and $q \in N$ a regular value, we define

$$\deg_q(f) = \sum_{p \in f^{-1}(q)} \varepsilon(p),$$

the sum of the signs of the values in $f^{-1}(q)$. We can also write this as

$$\deg_q(f) = \sum_{p \in f^{-1}(q)} \operatorname{sign}((Df)_p).$$

Note that this now defines the degree in $\mathbb{Z}$, not $\mathbb{Z}_2$.

We had two facts before about degree mod 2:

- $\deg(f)$ is independent of the choice of the regular value q
- $\deg(f)$ is equal for homotopic maps

Now we get the same thing in $\mathbb{Z}$ for M, N oriented.

Remember we had the cylinder $M \times I$ and we had inverse images of maps being loops that hung off the top or bottom, circles floating in the middle, or lines from the top to the bottom. But now we have orientation for all of them, and the degree on the boundary is 0, in the same way.

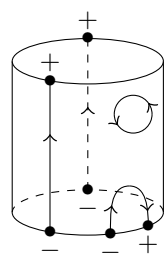

FIGURE 8.3.4.

8.4. Applications

8.4.1. The Fundamental Theorem of Algebra.

THEOREM 8.4.1 (Fundamental Theorem of Algebra). *Suppose f is a complex polynomial of algebraic degree > 0, then f has a root.*

PROOF. Extend $f : \mathbb{C} \to \mathbb{C}$ to the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$, using the coordinates on $S^2 \setminus \{\infty\} = \mathbb{C}$ the coordinate z; and on $S^2 \setminus \{0\}$ we use $1/z$.

So we extend f to $\hat{f} : S^2 \to S^2$ where $\hat{f}(z) = f(z)$, $z \neq \infty$, and $\hat{f}(\infty) = \infty$. This is continuous. We use the following lemma:

LEMMA 8.4.2. *The topological degree of $\hat{f}$ is equal to the algebraic degree of f.*

One way to see that they agree is that they are homotopy equivalent, so we reduce to a special case of replacing $\hat{f}$; in a special case $f(z) = z^n$ they obviously agree, then in general we can create a homotopy from a polynomial of degree n, say $z^n + a_1 z^{n-1} + \ldots + a_n$ to z^n, by $f_t(z) = z^n + (1-t)(a_1 z^{n-1} + \ldots + a_n)$. This yields by extension a homotopy of $\hat{f}(z)$ to $\widehat{(z^n)}$.

But if f has no root, the topological degree of f is 0, since 0 is a regular value, and then $\deg_0(\hat{f}) = 0$. $\square$

Note that this also immediately tells us why us have n roots, since in the inverse image of any regular value, we should get n values with sign. In fact $f(z) = c$ has n roots.

8.4.2. Homotopy Groups of Spheres. We look at applications of degree of maps to the $\pi_k(S^n)$, where $\pi_k(X) = [S^k, X]_+$, the *k-th homotopy group* of X, that is, the set of all homotopies of S^k into X. We studied in great detail the case of $k = 1$, which is the fundamental group. We will look briefly at $k > 1$.

What is non-trivial is that for $k > 1$, $\pi_k(X)$ turns out to be an abelian group.

One way to think about addition is if we have a map f from the k-sphere to X and another map g from the k-sphere to X, for $f + g$ we look at the quotient map $S^k \xrightarrow{q} S^k/S^{k-1}$ squeezing the equator to a point, so we will take $f + g = (f \wedge g) \circ q$ where $f \wedge g$ is taking f on the "top" sphere and g on the "bottom" sphere.

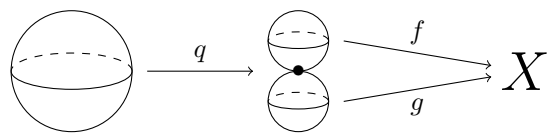

FIGURE 8.4.3.

Another way to think about the addition is as follows: maps $S^k \to X$ can be represented also by $D \xrightarrow{f} X$ with $f(S^{k-1}) = x$, where x is the basepoint. Then this is the same as mapping a k-sphere, since this is squeezing the boundary of D^k to the point. This is like in the case where $k = 1$ where we took a line and sent the two endpoints to a point. That is, we have $D^k \to D^k/S^{k-1} = S^k \to X$.

A third way is to consider maps $\mathbb{R}^k \xrightarrow{f} X$ with $f(\mathbb{R}^k \setminus \text{bounded set}) = x$. So if everything outside some bounded set gets squeezed to a point then this is the same thing.

The new idea is that $\pi_k(X)$ is commutative. Why is it commutative? There are many proofs of this; a very nice way to do it is to use adjoint functors and categories, but a very fast geometric proof is the following:

If $k > 1$, then everything in f except for a bounded section and everything in g except for a bounded section go to the basepoint, then we can just rotate to get $g + f$. This works in higher dimensions but not in $k = 1$ since in higher dimensions we have more degrees of freedom.

Another proof is, letting ΩX being the loops on X based at x, to show that $\pi_k(X) = \pi_{k-1}(\Omega X)$.

Well there is no general way to compute these groups, even for spheres, but we can show two facts: that $\pi_k(S^n) = 0$ for $0 < k < n$ and $\pi_n(S^n) \xrightarrow{\deg} \mathbb{Z}$ is an isomorphism.

Maps wrapping high-dimensional spheres to lower dimensional spheres are very tricky, and in fact quite surprising to hear exist. What is known is that $\pi_3(S^2) = \mathbb{Z}$, and in fact one can show that in general for $m > n$, $\pi_n(S^m)$ is a finite abelian group, except for when m is even and $n = 2m - 1$, in which case it is $\mathbb{Z} \oplus$ finite group. However their orders are quite complicated, and their computation involves a lot of number theory and algebra.

THEOREM 8.4.4. *$\pi_k(S^n) = 0$ for $0 < k < n$.*

PROOF. So for $S^k \xrightarrow{f} S^n$, we saw that $f \underset{h}{\sim}$ smooth map, so we can assume up to homotopy that f is smooth. Now most points were regular values; they are

open and dense in S^n. But if $k < n$, the only regular values are the ones not in the image, that is, "most" points are not in the image of f. So pick $q \notin \mathrm{Im}(f)$. But then we are done since once we have $S^k \to S^n \setminus \{q\}$, well $S^n \setminus \{q\}$ is $\mathbb{R}^n$, which is contractible! So f is nullhomotopic. $\square$

THEOREM 8.4.5. *$\pi_n(S^n) \overset{\deg}{\to} \mathbb{Z}$ is an isomorphism.*

PROOF. As we saw previously, we had a well-defined map $\pi_n(S^n) \overset{\deg}{\to} \mathbb{Z}$. Now it is easy to see that $\deg(f+g) = \deg(f) + \deg(g)$: we can just use smooth f and g and just count. Notice that $\deg(I_{S^n}) = 1$. So deg is surjective.

EXERCISE 8.4.6. Check that $f + (-f) \underset{h}{\sim} 0$.

The tricky thing is to show that $\deg : \pi_n(S^n) \to \mathbb{Z}$ is injective.

The case $n = 1$ we did, $\pi_1(S^1) = \mathbb{Z}$ generated by $[\mathrm{Id}_{S^1}]$.

In general there is a homomorphism of suspension $\pi_k(X) \to \pi_{k+1}(\Sigma X)$. We claim that there is a way to compare maps in one dimension to maps in one dimension higher. In fact in general for $A \overset{f}{\to} B$ we can get $\Sigma A \overset{\Sigma f}{\to} \Sigma B$ where for the suspension map Σf we just take f at every "level", that is, for $[-1,1] \times A \to [-1,1] \times B$ we use $\Sigma f = \mathrm{Id}_{[-1,1]} \times f$ and quotient out at -1 and 1. In particular for spheres we have $\pi_k(X) \overset{\Sigma}{\to} \pi_{k+1}(\Sigma X)$. It is easy to see that this is a homomorphism and is compatible with homotopy.

Now why is this of interest? We can try to use this since we had $\pi_1(S^1) \overset{\deg}{\to} \mathbb{Z}$, and we have $\pi_1(S^1) \overset{\Sigma}{\to} \pi_2(S^2) \overset{\Sigma}{\to} \pi_3(S^3) \overset{\Sigma}{\to} \dots$:

$$\begin{tikzcd}
\pi_1(S^1) \arrow[r, "\Sigma"] \arrow[dr, "\deg"'] & \pi_2(S^3) \arrow[r, "\Sigma"] \arrow[d, "\deg"] & \pi_2(S^3) \arrow[r, "\Sigma"] \arrow[dl, "\deg"] & \cdots \\
& \mathbb{Z} & &
\end{tikzcd}$$

FIGURE 8.4.7.

We wanted to show that all of the degree maps from $\pi_k(S^k) \to \mathbb{Z}$ are also isomorphisms. Now $\deg(\Sigma f) = \deg(f)$; the proof is to take f smooth and count: at any level the degree is the same. So this is another way to see that the degree is surjective.

We see that $\pi_k(S^k) = \mathbb{Z} \oplus ?$ where $\mathbb{Z}$ is just the multiples of $[\mathrm{Id}_{S^k}]$. Now we want to see that the ? is 0. More concretely the ? is the ker(deg). We need to see that it isn't there. Well in $k = 1$ we already saw that, so it's alright.

For the rest, we will argue by induction. The inductive step will be the following lemma:

LEMMA 8.4.8. *$\Sigma : \pi_k(S^k) \to \pi_{k+1}(S^{k+1})$ is surjective.*

Once we know this we will be done, since then $\pi_2(S^2)$ is just $\mathbb{Z}$, and so on, since we already knew that for $k > 1$ that $\pi_k(S^k)$ contains $\mathbb{Z}$ from the suspension. This yields inductively that $\pi_k(S^k) = \mathbb{Z}$.

The fast way is to show that every map in $\pi_{k+1}(S^{k+1})$ can be desuspended. In general there is no way to desuspend a map.

So given $g : S^{k+1} : S^{k+1}$ we will show that $g \underset{h}{\sim} \Sigma f$, $f : S^k \to S^k$. So pick a p, q regular values of g, where q is in the upper hemisphere and p is in the lower hemisphere, then consider $g^{-1}(q)$ and $g^{-1}(p)$. Now by uniformity of manifolds we can assume that $g^{-1}(q)$ is in the upper hemisphere and $g^{-1}(p)$ is in the lower hemisphere. Now if we take a little disk around q and a little disk around p, which are sort of like polar caps, then squeeze all but the caps around q and p to the equator. So we can thus assume that the equator S^k is mapping to S^k, the lower hemisphere to maps to the lower hemisphere, and the upper hemisphere maps to the upper hemisphere.

In summary, any map $g : S^{k+1} \to S^{k+1}$ is homotopic to a map that sends the the upper hemisphere to the upper hemisphere, the lower hemisphere to the lower hemisphere, and the equator to the equator. Then $g \underset{h}{\sim} \Sigma(g|_{\text{equator}})$ where $g|_{\text{equator}} : S^k \to S^k$: we linearly interpolate between g and $\Sigma(g|_{\text{equator}})$ on the upper disk and the lower disk. Notice that nothing changes on the equator. $\square$

There is a deep theorem about desuspensions in general:

Theorem 8.4.9 (Freudenthal Desuspension Theorem). *$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X)$ is isomorphic if X is n-connected and $k < 2n - 1$.*

A space is n-connected if it is connected and $\pi_k(X) = 0$ for $k \leqslant n$.

What this tells us is that after a while all of the maps

$$\pi_3(S^2) \xrightarrow{\Sigma} \pi_4(S^3) \xrightarrow{\Sigma} \pi_5(S^4) \xrightarrow{\Sigma} \cdots$$

are isomorphisms, that is, it stabilizes to the right to $\mathbb{Z}_2$. There is a field studying this called stable homotopy theory.

Part IV

Homology

CHAPTER 9

Homology Groups

9.1. Cellular Homology

9.1.1. Cell Complexes. The idea is that we build spaces inductively using disks of increasing dimension.

We build up a space X by starting with some 0-dimensional space sitting in some 1-dimensional space, etc, $X^0 \subset X^1 \subset \cdots \subset X^n = X$, where X^i is the i-skeleton.

X^0 is just a set of points with the discrete topology. X^1 is a graph. In general X^{k+1} is obtained as follows: take X^k and glue on stuff of the next dimension, in general disks which we call e, so $X^{k+1} = X^k \cup e^{k+1} \cup e^{k+1} \cup \ldots$, where these $k+1$ dimensional disks are glued to X^k using what are called attaching maps.

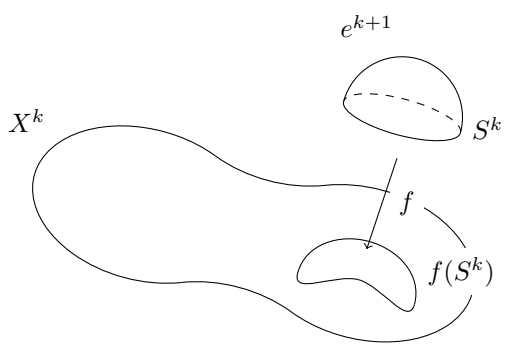

FIGURE 9.1.1.

These are maps $f : S^k = \partial e^{k+1} \to X^k$, then we glue by taking the quotient $X^k \cup e^{k+1}/(u \sim f(u))$ where $u \in S^k = \partial e^{k+1}$ and $f(u) \in X^k$.

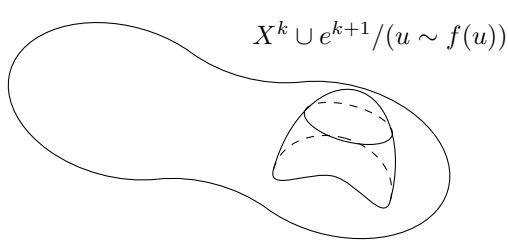

FIGURE 9.1.2.

Example 9.1.3. One way to form a circle is to take two points, and then glue two lines to it. That is, $S^1 = X$ has $X^0 =$2 points, then $X^1 = 2$ points $\cup_f e^1 \cup_g e^1$ where f and g attach one end of the line to one point and the other end to the other point.

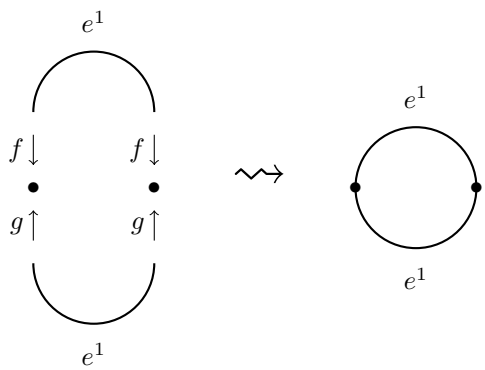

Figure 9.1.4.

Another way is to break it up into three points and three edges, or we can even just use one point and one 1-cell, with both ends squeezed to a point.

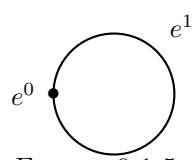

Figure 9.1.5.

If the attaching map is required to be an inclusion then this is called a regular complex.

In general we want as few complexes as possible since that allows the eventual algebra to be very simple.

Example 9.1.6. $S^2 = e^0 \cup e^0 \cup_\alpha e^1 \cup_\beta e^1 \cup_f e^2 \cup_g e^2$.

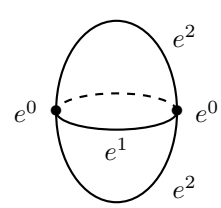

Figure 9.1.7.

This can be extended into a cell decomposition of S^n. This is a very expensive way to construct S^n, but it is a regular decomposition. Another way to construct S^n is $S^n = e^0 \cup_f e^n$ where f sends S^{k-1} to a point.

Example 9.1.8. In $\mathbb{R}\mathrm{P}^n$ has all of the antipodes identified, so $\mathbb{R}\mathrm{P}^0 = e^0$, $\mathbb{R}\mathrm{P}^1 = e^0 \cup_{f_0} e^1$, ..., $\mathbb{R}\mathrm{P}^n = e^0 \cup_{f_0} e^1 \cup_{f_1} e^2 \cup_{f_2} \ldots \cup_{f_{n-1}} e^n$. This has a nice inclusion since each $\mathbb{R}\mathrm{P}^k$ attaches to the previous one $\mathbb{R}\mathrm{P}^{k-1}$. The attaching map $\partial e^n = S^{n-1} \overset{f_{n-1}}{\to} \mathbb{R}\mathrm{P}^{n-1}$ is the 2-to-1 covering map wrapping around twice. It turns out that this is a minimal cell decomposition for $\mathbb{R}\mathrm{P}^n$.

Example 9.1.9. One way to decompose a torus is to take a 0-dimensional cell e^0 and attaching two edges e^1_a and e^1_b so that it looks like a figure 8, then everything else is just e^2 filling everything in.

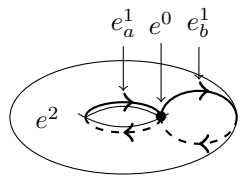

Figure 9.1.10.

Another way to look at this is as follows: recall that we can get a torus by taking a square and gluing opposite sides together. Now when we do this all the corners get identified to the same point, so this is e^0. The horizontal edges are e^1_a and the vertical edges are e^1_b. Then the inside is e^2.

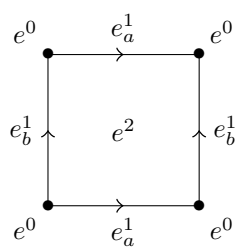

Figure 9.1.11.

So we get Torus $= e^0 \cup_\alpha e^1 \cup_\beta e^1 \cup_f e^2$, where α and β are obvious, and then $f = e^1_a e^1_b (e^1_a)^{-1} (e^1_b)^{-1}$.

Exercise 9.1.12. Do the same thing for the Klein bottle.

In fact one can show that every smooth manifold has a cell decomposition, but these need not be unique. In fact not even the minimal one is unique.

Cell decompositions are a very useful auxiliary tool, but we will need to confront the problem of showing that what we get from the cell decomposition is independent of the decomposition itself.

9.1.2. Using Cell Decompositions. What we want to do is "count" k-dimensional holes in X. Now what is a hole? It seems like a circle has a hole in it. Intuitively, we can define a k-dimensional hole as a k-dimensional area with the property that with boundary equal to 0. But this is not quite correct, since D^2 is like a circle but filled in. So we have to throw out the ones that are filled in. So

we quotient out by the boundaries of $k+1$ dimensional areas. When we make this precise we will have the homology group $H_k(X)$.

So let $C_k(X)$, called the k-dimensional chains of X, be the sum of k-cells with coefficients for the moment in the integers. So $C_k(X) = \bigoplus_{k\text{-cells in } X} \mathbb{Z}$.

EXAMPLE 9.1.13. For a torus, we had a decomposition Torus $= e^0 \cup e^1_a \cup e^1_b \cup e^2$. So $C_2(X) = \mathbb{Z}e^2$, $C_1(X) = \mathbb{Z}e^1_a \oplus \mathbb{Z}e^1_b$, and $C_0(X) = \mathbb{Z}e^0$.

Now these chain groups depend heavily on the decomposition. But between each of these will be a boundary map ∂_k where k is the dimension. Without a formal definition, what is going to be the boundary of the 2-cell in the torus? Well, $\partial e^2 = e^1_a + e^1_b - e^1_a - e^1_b = 0$. What about the 1-cells? Well $\partial e^1 = e^0 - e^0 = 0$ as well.

To make things more precise,

$$H_k(X) = \ker(C_k(X) \overset{\partial_k}{\to} C_{k-1}(X)) / \operatorname{Im}(C_{k+1}(X) \overset{\partial_{k+1}}{\to} C_k(X))$$

LEMMA 9.1.14. *When $\partial_k = 0$ and $\partial_{k+1} = 0$ then $H_k(X) = C_k(X)$.*

So for a torus, the homology groups are the chain groups. That is, $H_2(\text{torus}) = \mathbb{Z}$, $H_1(\text{torus}) = \mathbb{Z} \oplus \mathbb{Z}$, and $H_0(\text{torus}) = \mathbb{Z}$. It turns out that H_0 measures how many pieces there are, so for the torus there is only one piece.

The amazing fact is that if we took a different decomposition, we get different chain groups but the homology groups are the same.

A big thing that we will not prove is the following:

THEOREM 9.1.15. *The $H_k(X)$ are independent of the choice of a cell decomposition of X.*

So these homology groups are a very important way of representing the holes in a space.

So before even giving formal definitions of boundary maps, we can assign some exercises:

EXERCISE 9.1.16. Compute $H_k(S^2)$.

EXERCISE 9.1.17. Compute $H_k(\mathbb{R}\mathrm{P}^2)$.

Now the coefficients $\mathbb{Z}$ here can be replaced with, say, $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Z}_2$, then these groups would be for example $H_2(\text{torus}) = \mathbb{Z}_2$, $H_1(\text{torus}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H_0(\text{torus}) = \mathbb{Z}_2$. In fact $\mathbb{Z}$ already carries the most information, but there are reasons to use $\mathbb{R}$ or $\mathbb{C}$ to fit well in the system being used.

9.1.3. Chain Complexes and Cellular Homology. We make rigorous going from the cell complex X to the *cellular chain complex*

$$C_n(X) \overset{\partial_n}{\to} C_{n-1}(X) \overset{\partial_{n-1}}{\to} C_{n-2}(X) \overset{\partial_{n-2}}{\to} \cdots,$$

as well as the *cellular homology groups*

$$H_k(X) = \frac{\operatorname{Ker}(C_k(X) \overset{\partial_k}{\to} C_{k-1}(X))}{\operatorname{Im}(C_{k+1}(X) \overset{\partial_{k+1}}{\to} C_k(X))}.$$

Previously we were able to see when the boundary maps were 0 geometrically, but we need a general way to do this.

Formally, we will have $C_n(X;R) = \bigoplus_\alpha R_\alpha$ where α is an n-cell of X, and R is the field of coefficients $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Z}_2, \mathbb{Z}_p$ depending on what is useful. One of the nice things about using $\mathbb{Z}_2$ is that one can ignore signs.

We need a definition of ∂_n, the linear map from $C_n(X;R)$ to $C_{n-1}(X;R)$, which records how n-cells are attached to $(n-1)$-cells.

Suppose the n-cells of X are given by $e_1^n, \ldots, e_k^n$, and the $(n-1)$-cells of X are given by $e_1^{n-1}, \ldots, e_\ell^{n-1}$, then ∂_n is a $(k \times \ell)$-matrix. Then what are the entries of the matrix? Well it will be geometrically obvious when we work it out. Well we had the $(n-1)$-skeleton X^{n-1}, then we have an attaching map $\partial e_i^n = S_i^{n-1} \to X^{n-1}$, where $X^n = X^{n-1} \cup_{f_1} e_1^n \cup_{f_2} e_2^n \cup \ldots \cup_{f_k} e_k^n$, and $X^{n-1} = X^{n-2} \cup e_1^{n-1} \cup \ldots \cup e_\ell^{n-1}$.

Now in the matrix $\partial_n = ((a_{ij}))$, then a_{ij} records how many times the boundary of $S_i^{n-1} = \partial e_i^n$ "runs through" the cell e_j^{n-1}.

Now let us look at what happens when we take X^{n-1} and we squeeze out the $(n-2)$-skeleton to get X^{n-1}/X^{n-2}. Now what do we get? Each of these disks e_i^{n-1} becomes a sphere, so we get a nice bouquet of spheres. Now suppose we further divided by everything except the single sphere,

$$X^{n-1}/(X^{n-2} \cup e_1^{n-1} \cup \ldots \cup e_{j-1}^{n-1} \cup e_{j+1}^{n-1} \cup \ldots \cup e_\ell^{n-1}) = S_j^{n-1}.$$

Call this quotient map q_j since it focuses our attention on the j-th sphere. So now a_{ij} should determine how many times S_i^{n-1} runs around S_j^{n-1} via $g_j \circ f_i$. So how do we do this? We just use the degree, so $a_{ij} = \deg(q_j \circ f_i)$. And that is the definition of ∂_n.

Example 9.1.18. Let X be the Klein bottle, which is a square with its sides identified, then we have $X = e_0 \cup e_a^1 \cup e_b^1 \cup e^2$ where e_b^1 is the two sides identified with a flip:

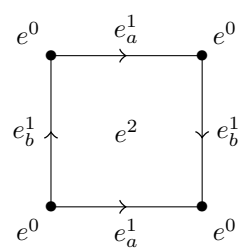

Figure 9.1.19.

Then $C_2(X) = Re^2$, $C_1(X) = Re_a^1 \oplus Re_b^1$, and $C_0(X) = Re^0$. Well $\partial_1 = 0$, but ∂_2 is a bit more tricky. Well $\partial_2(e^2) = e_a^1 + e_b^1 - e_a^1 + e_b^1 = 0e_a^1 + 2e_b^1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Now let us compute the homology groups. Of course for different coefficients we will get different numbers, but let us do it for $R = \mathbb{Z}$. Well for dimension 0, everything is in the kernel, and the image is 0, so $H_0(X) = \mathbb{Z}/0 = \mathbb{Z}$. For dimension 1, this is tricky because the kernel is $\mathbb{Z} \oplus \mathbb{Z}$, but we are dividing out by the image of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, so we have $H_1(X) = (\mathbb{Z} \oplus \mathbb{Z})/(0 \oplus 2\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$. Now for dimension 2, the kernel of ∂_2 is 0, and the image from above is 0, so we have $H_2(X) = 0/0 = 0$.

Now let us take $R = \mathbb{Q}$. For dimension 0, we'd just get $\mathbb{Q}$. But for dimension 1, we would get $(\mathbb{Q} \oplus \mathbb{Q})/(0 \oplus 2\mathbb{Q}) = \mathbb{Q}$, since everything in $\mathbb{Q}$ is a multiple of 2. This is usual for fields of characteristic 0. For dimension 2 we get 0 again as well.

Now for $R = \mathbb{Z}_2$, then in dimension 0, it's $\mathbb{Z}_2$, and in dimension 1 we get $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, but what happens in dimension 2? Well $\partial_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so the kernel is everything, and we get $\mathbb{Z}_2$! Interestingly enough, using $R = \mathbb{Z}_2$ we get the same homology groups for the Torus, since we no longer care about signs.

Now we have snuck something in here, that we have not objected to before. We had $H_k(X) = \operatorname{Ker}(\partial_k)/\operatorname{Im}(\partial_{k+1})$. In order to take this quotient we need to know that $\operatorname{Im}(\partial_{k+1}) \subset \operatorname{Ker}(\partial_k)$. Well it turns out that it is, and we have been implicitly using it, but we did not prove it before. This is equivalent to saying that $\partial_k \circ \partial_{k+1} = 0$.

So now a *chain complex*, more precisely, is a sequence of abelian groups (or vector spaces in particular) with linear maps that can be arranged

$$C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} C_{n-2} \overset{\partial_{n-2}}{\to} \cdots \overset{\partial_1}{\to} C_0$$

satisfying $\partial_k \circ \partial_{k+1} = 0$.

In fact we will not prove that $\partial_k \circ \partial_{k+1} = 0$ here, since we will see another setting, with another definition of the chain complex, where it is easier to prove.

As an aside, we note that $H_k = 0 \iff \operatorname{Im} \partial_{k+1} = \operatorname{Ker} \partial_k$. We then say that the chain complex is exact at k, and the entire chain is called an *exact sequence* if it is exact at all k. That is, this is a chain complex whose homology vanishes. $\operatorname{Ker} \partial_k$ is sometimes known as the group of k cycles written Z_k, and $\operatorname{Im} \partial_{k+1}$ is the k-boundaries B_k. So $H_k = Z_k/B_k$.

9.2. Simplicial Homology

Whereas before we cut spaces up into cell complexes, now we cut things up into triangles.

For two dimensions, a good way to get an equilateral trinagle is to, in three dimensions, take points at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ and join them.

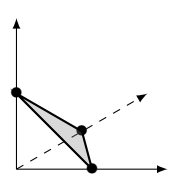

FIGURE 9.2.1.

DEFINITION 9.2.2. The n-dimensional *simplex* (or n-simplex) Δ^n is a set of vertices $\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid 0 \leqslant x_i, \sum x_i = 1\}$.

For example, Δ^1 is just a line. By deleting x_0, we can get an injective map $f_n : \Delta^n \to \mathbb{R}^n$ such that $f_n(\Delta^n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leqslant x_i; \sum x_i \leqslant 1\}$. This gives us another way to view the n-simplex.

There are points that are vertices of Δ^n, which we will write as $v_0 = (1, 0, \ldots, 0)$, $\ldots$, $v_n = (0, 0, \ldots, 1)$. So we may write $\Delta^n = \langle v_0, \ldots, v_n \rangle$.

Now a face of a simplex $\langle v_0, \ldots, v_n \rangle$ is obtained by deleting one of the vertices. Now we define the boundary of a simplex a sum of combinations of its simplices. Now we need to introduce signs, since even in one-dimension we have $\partial\langle v_0, v_1\rangle = v_1 - v_0$. So $\partial_n(\Delta^n) = \sum_{i=0}^{n}(-1)^i\langle v_0, v_1, \ldots, \hat{v_i}, \ldots, v_n\rangle$ where $\hat{v_i}$ denotes that v_i is removed. Then for a triangle $\langle v_0, v_1, v_2\rangle$ going counterclockwise as in Figure 9.2.3, we have $\partial\langle v_0, v_1, v_2\rangle = \langle v_0, v_1\rangle + \langle v_1, v_2\rangle - \langle v_0, v_2\rangle$.

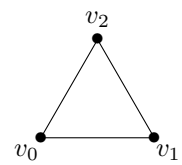

FIGURE 9.2.3.

Now how do we use this? We take any space, cut it up into triangles, and then get a chain complex and then compute the homology groups. This is purely mechanical and does not require degree of maps.

So a triangulation of X is a covering of X by 1-to-1 images of order-preserving maps $\langle v_0, \ldots, v_n\rangle \xrightarrow{f} X$, which we call *simplices* on X satisfying the following conditions:

(1) Any face of a simplex on X is also a simplex in X.
(2) Any two simplices intersect only in one of their subsimplices, or all of its subsimplices.

EXAMPLE 9.2.4. We get a traingulation of the torus. Recall that we can just get a torus by taking a square and identifying its sides. Now we can cut up the square into ninths, then draw a diagonal across each little square from bottom left to top right.

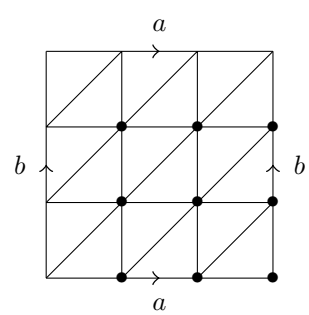

FIGURE 9.2.5.

The number of Δ^2 simplices is 18, the number of Δ^1 simplices is 27, and the number of Δ^0 simplices is 9.

So we can now form what is called the *simplical chain complex* of a space X. As before, we let $C_k(X)$ be the formal linear combinations of the Δ^k in X. Then we can get $C_k(X) \xrightarrow{\partial_k} C_{k-1}(X)$ using the previously given formula, and then define

homology H_k as before, which will turn out to be isomorphic to the homology groups as in cellular chain complexes.

We will prove the following:

PROPOSITION 9.2.6. $\partial_{k-1} \circ \partial_k = 0$.

PROOF. $\partial_{k+1} \circ \partial_k(\langle v_0, \ldots, v_k \rangle) = \sum (\pm 1 \mp 1)^i \langle v_0, \ldots \hat{\cdot} \ldots, \ldots \hat{\cdot} \ldots, v_k \rangle$.

EXERCISE 9.2.7. Check that the $(-1)^i$ always come up with opposite signs, regardless of which vertices are ommitted.

We will check it for the two-dimensional case:

$$\partial_2(\langle v_0, v_1, v_2 \rangle) = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle$$

then

$$\partial_1 \circ \partial_2(\langle v_0, v_1, v_2 \rangle) = (v_1 - v_0) + (v_2 - v_1) - (v_2 - v_0) = 0 \qquad \square$$

People attempted to prove that the homology groups are independent of the triangulation, since it is easily shown that refining the triangulation by cutting up the triangle even more does not change the homology groups, then if one could show that any two triangulations had a common refinement, this would be done. It turns out that this is not true.

9.3. Singular Homology

Before we had Cell Complexes which led to cellular chain complexes, and then we had Triangulation which led to simplicial chain complexes, and in both cases we computed the homology groups $H_*(X) = \frac{\mathrm{Ker}(\partial_*)}{\mathrm{Im}(\partial_{*+1})}$.

Now we will take a space X and get a singular chain complex, and then we get the homology groups in the same way.

So a *singular*[1] *k-simplex in X* is a function $f : \Delta^k \to X$, so it is a parameterized k-simplex into X. We allow f not injective, so this allows very crummy images. So in other words, we will allow the image of Δ^k to cross itself.

Then we define the *singular chain complex* $C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \to \cdots$, where $C_k(X)$ is the finite formal linear combination of all singular k-simplices.

EXAMPLE 9.3.1. We look at the case of $X = \{\text{point}\}$. We will get something wacky where we get something in every dimension, but hopefully when we get to homology it all wipes out. Well for $\Delta^k \to \{\text{point}\}$ there is only one map, the map sending the complex to a point. Using coefficients $R = \mathbb{Z}$, then $C_*(\text{point}) = \mathbb{Z}$ in every dimension $\geqslant 0$. So the chain complex is very big:

$$\cdots \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}.$$

But the boundary is defined in the usual way, as the alternating sum of its faces, regarded using the restrictions of $f : \Delta^k \to X$ as singular $(k-1)$-simplices in X.

Well $\partial_1(\Delta^1) = \Delta^0 - \Delta^0 = 0\Delta^0$, but $\partial_2(\Delta^2) = \Delta^1 - \Delta^1 + \Delta^1 = 1\Delta^1$, and so on. So in the chain complex of a point, the ∂_i are alternately $+1$ and 0. To check, we note that $\partial_{k-1}\partial_k = 0$, as one of them is 0. So this is as we expect a chain complex.

So that is the chain complex, but what is the homology, which we really care about? There are only three cases we care about: dimension 0, and then odd and even dimensions, since the chain complex repeats.

[1]This means that it is allowed to be singular, not that it must be

Well $H_0 = \frac{\mathrm{Ker}(C_0(X)\to 0)}{\mathrm{Im}(C_1(X)\xrightarrow{\partial_1} C_0(X))} = \frac{\mathrm{Ker}(\mathbb{Z}\to 0)}{\mathrm{Im}(\mathbb{Z}\xrightarrow{\times 0}\mathbb{Z})} = \frac{\mathbb{Z}}{0} = \mathbb{Z}$.

Next, $H_1(X) = \frac{\mathrm{Ker}(\mathbb{Z}\xrightarrow{\partial_1=0}\mathbb{Z})}{\mathrm{Im}(\mathbb{Z}\xrightarrow{\partial_2=1}\mathbb{Z})} = \frac{\mathbb{Z}}{\mathbb{Z}} = 0$. This is also $H_{\mathrm{odd}}(X)$.

Finally, $H_{\mathrm{even}}(X) = H_2(X) = \frac{\mathrm{Ker}(\mathbb{Z}\xrightarrow{1}\mathbb{Z})}{\mathrm{Im}(\mathbb{Z}\xrightarrow{0}\mathbb{Z})} = 0$. So then $H_0(\text{point}) = \mathbb{Z}$ and $H_*(\text{point}) = 0$ for $* \neq 0$.

So even though the chain complex looked dangerous, it all washes out and ends up not mattering at the end of the day. We can show this in general.

The disadvantage of this is that it gives an uncountable basis to work with, but it has a few big advantages. One is that no extra structure on X is being used (that is, this is choiceless). The second is that it makes it easier to compare spaces.

CHAPTER 10

Comparing Homology Groups of Spaces

10.1. Induced Homomorphisms

Recall that when we had $X \xrightarrow{f} Y$ we had induced maps on $\pi_1, \ldots, \pi_n, \ldots$. We want induced maps in $H_*(\cdot)$, so that we can compare homology groups via continuous maps between them.

Example 10.1.1. A special case is suppose that $A \underset{i}{\hookrightarrow} X$, and in fact A is a sub-cell complex of X. It is very obvious how we are going to go around comparing their chain complexes: $C_k(A) \underset{i_k}{\hookrightarrow} C_k(X)$. Similarly, we get $C_{k-1}(A) \underset{i_{k-1}}{\hookrightarrow} C_{k-1}(X)$. So not only do we get the corresponding chain complexes, the boundary maps are in fact compatible, and we get a commutative diagram:

$$\begin{tikzcd}
C_k(A) \arrow[r, hook, "i_k"'] \arrow[d, "\partial_k"] & C_k(X) \arrow[d, "\partial_k"] \\
C_{k-1}(A) \arrow[r, hook, "i_{k-1}"'] & C_{k-1}(X)
\end{tikzcd}$$

Figure 10.1.2.

That is, $\partial_k \circ i_k = i_{k-1} \circ \partial_k$.

So the story is that the whole of the chain complex of A is sitting in the chain complex of X. For short, we will say that $C_*(A) \underset{(i_k)}{\hookrightarrow} C_*(X)$.

That is fine on the level of chain complexes, but what we really care about is homology. So what are the consequences for homology?

It is easy to see that as an algebraic consequence this implies that

$$\mathrm{Ker}(C_k(A) \xrightarrow{\partial_k} C_{k-1}(A)) \underset{i}{\hookrightarrow} \mathrm{Ker}(C_k(X) \xrightarrow{\partial_k} C_{k-1}(X)).$$

This is called a diagram chase argument: we just follow the linear maps around. In detail, if $z \in \mathrm{Ker}(C_k(A) \xrightarrow{\partial_k} C_{k-1}(A))$ then by definition $\partial_k(z) = 0$ in $C_{k-1}(A)$, so, $i_{k-1} \circ \partial_k(z) = 0$ in $C_{k-1}(X)$. But this is $\partial_k \circ i_k(z) = i_{k-1} \circ \partial_k(z) = 0$. So $i_k(z) \in C_k(X)$ is in $\mathrm{Ker}(C_k(X) \xrightarrow{\partial_k} C_{k-1}(X))$.

Exercise 10.1.3. Similarly, show that

$$\mathrm{Im}(C_k(A) \xrightarrow{\partial_k} C_{k-1}(A)) \xrightarrow{i_k} \mathrm{Im}(C_k(X) \xrightarrow{\partial_k} C_{k-1}(X)).$$

So if kernels map to kernels and images map to images, then quotients map to quotients: $H_k = \frac{\mathrm{Ker}}{\mathrm{Im}}$ maps as well. So we get a map, called the induced map

or induced homomorphism, from $H_k(A) \xrightarrow{i_*} H_k(X)$. This maps the k-dimensional holes in A to their image in X under inclusion.

So we see that if A is a cellular complex X we get an induced homomorphism $H_k(A) \to H_k(X)$.

A similar argument works for triangulated spaces (simplicial complexes).

A warning: even though $A \underset{i}{\hookrightarrow} X$ is an inclusion and similarly for the chain complexes, the kernels, and the images, the map $H_k(A) \to H_k(X)$ may fail to be injective. The algebraic reason is that this involves quotient groups. Geometrically, just because A is sitting in X does not mean X has more holes than A; some holes in A may get filled in X. So intuitively something might go to 0. We will see that this is what really happens.

Example 10.1.4. Take the decomposition $S^n = e^0 \cup_f e^n$ where $f : S^{n-1} \to e^0$ is a constant map. That is a very simple picture since it only has cells in dimension 0 and n, so the chain complex has $\mathbb{Z}$ in dimension 0 and $\mathbb{Z}$ in dimension n, and 0 everywhere else. So obviously all of the boundary maps are 0, so that $H_i \cong C_i$, and we have

$$H_*(S^n) = \begin{cases} \mathbb{Z} & * = 0 \text{ or } * = n \\ 0 & 0 < * < n \end{cases}.$$

In a disk, we should see the homology disappear. So let us do a cell decomposition for a disk: $D^{n+1} = e^0 \cup_f e^n \cup_g e^{n+1}$ where f is as before and $g : S^n \to S^n$ is the identiy map Id_{S^n}, where e^{n+1} fills in the sphere. So now the picture is $\mathbb{Z}$ in dimension 0, then there is $\mathbb{Z}$ in dimension n, and another $\mathbb{Z}$ in dimension $(n+1)$. So $\partial_{n+1} = \times 1 = \deg(g)$. So for the homology what is new is what happens in dimension n:

$$H_n(D^{n+1}) = \frac{\mathrm{Ker}(\mathbb{Z} \to 0)}{\mathrm{Im}(\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z})} = \frac{\mathbb{Z}}{\mathbb{Z}} = 0.$$

But

$$H_{n+1}(D^{n+1}) = \frac{\mathrm{Ker}(\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z})}{\mathrm{Im}(0 \to \mathbb{Z})} = \frac{0}{0} = 0$$

so

$$H_*(D^{n+1}) = \begin{cases} 0 & * > 0 \\ \mathbb{Z} & * = 0 \end{cases},$$

which is the same as a point.

Now what happens when we compare the sphere and the disk? Well we have $S^n \overset{i}{\hookrightarrow} D^{n+1}$, but $\mathbb{Z} = H_n(S^n) \xrightarrow{i_* = 0} H_n(D^{n+1}) = 0$, so the induced map is not injective.

This tells us what to do for inclusions, but the question is how to do this for any continuous map.

More generally, given any continuous map $f : X \to Y$ we get induced homomorphisms $f_* : H_k(X) \to H_k(Y)$, with any coefficients. So far we considered the case where f is an inclusion.

This is most easily seen in singular homology. The reason is the following: for a parameterized (singular) simplex $\Delta^k \xrightarrow{\alpha} X$, and we take it under a map $X \xrightarrow{f} Y$, it is easy to see how to get a parameterized simplex in Y: just take the composite map. We do not have a problem if the image $f \circ \alpha$ is not nice. So $f_*(\alpha) = (f \circ \alpha)$

gives a map on bases of singular simplices and thus $C_*(X) \to C_*(Y)$ just as before for inclusions, so that we get a commutative map with the boundary maps, etc. So we get $H_*(X) \xrightarrow{f_*} H_*(Y)$.

To break this up into steps, given $X \xrightarrow{f} Y$, we get a homomorphism of the singular chain group in dimension k, $C_k(X) \xrightarrow{f_k} C_k(Y)$. We defined a basis element, a parameterized k-simplex in X, $\Delta^k \xrightarrow{\alpha} X$ by $f_k(\alpha) = (f \circ \alpha)$, which is a basis element in $C_k(Y)$.

$$\begin{tikzcd}
 & \arrow[d, "\partial^X_{k+1}"] & & \arrow[d, "\partial^Y_{k+1}"] \\
 & C_k(X) \arrow[r, "f_k"] \arrow[d, "\partial^X_k"] & & C_k(Y) \arrow[d, "\partial^Y_k"] \\
 & C_{k-1}(X) \arrow[r, "f_{k-1}"] \arrow[d, "\partial^X_{k-1}"] & & C_{k-1}(Y) \arrow[d, "\partial^Y_{k-1}"] \\
 & C_{k-2}(X) \arrow[r, "f_{k-2}"] \arrow[d, "\partial^X_{k-2}"] & & C_{k-2}(Y) \arrow[d, "\partial^Y_{k-2}"] \\
 & & &
\end{tikzcd}$$

FIGURE 10.1.5.

Then we have maps $\partial^Y_k \circ f_k = f_{k-1} \circ \partial^X_k$ for the chain complexes, and then we have $\mathrm{Ker}(\partial^X_k) \xrightarrow{f_k|} \mathrm{Ker}(\partial^Y_k)$ and $\mathrm{Im}(\partial^X_k) \xrightarrow{f_k|} \mathrm{Im}(\partial^Y_k)$, so the quotient homology groups have a homomorphism $H_k(X) \xrightarrow{f_*} H_k(Y)$.

It is a lot harder to do this for the other approaches (cellular or simplicial homology). A difficulty is that f may not be very compatible with the cell decomposition or the triangulation. There is a technique for doing this, by allowing subdivision of the cells (or simplices) and prove something called cellular (or simplicial) approximation, which says that if we make the cells (or triangles) fine enough, we can approximate the map by one that maps cells to cells (or triangles to triangles).

Some properties of the induced homomorphism are as follows: for $X \xrightarrow{\mathrm{Id}_X} X$, $(\mathrm{Id}_X)_* = \mathrm{Id}_{H_*(X)}$; and $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)_* = g_* \circ f_*$. These properties are called naturality, and this is similar to the properties of the induced homomorphism for the fundamental group.

EXERCISE 10.1.6.

(1) Show that if $A \underset{i}{\hookrightarrow} X$ has a retraction, that is, a map $r : X \to A$ with $r \circ i = \mathrm{Id}_A$, then $i_* : H_k(A) \to H_k(X)$ is injective.
(2) Show that there is no retraction from a D^{n+1} to S^n.
(3) Conclude the Brouwer Fixed Point Theorem.

The relation between $H_*(A)$ and $H_*(X)$ for $A \hookrightarrow X$ is subtle. We want to figure out what the story is.

To explore this, we introduce the *relative homology groups* $H_k(X, A)$ of the pair (X, A) for $A \subset X$. The idea is that this measures their "difference".

Now $C_k(A) \hookrightarrow C_k(X)$, so as an algebraist how do we compare them? We take their quotient. So we define the *relative chain group* $C_k(X, A) = C_k(X)/C_k(A)$. This works in all cases (cellular, simplicial, singular). These themselves form a relative chain complex of (X, A). Now we have the following diagram:

$$\begin{tikzcd}
C_k(A) \arrow[r, hook, "i_k"'] \arrow[d, "\partial_k^A"] & C_k(X) \arrow[r, "\text{quotient}"] \arrow[d, "\partial_k^X"] & C_k(X,A) \arrow[d, "\partial_k^{(X,A)}"] \\
C_{k-1}(A) \arrow[r, hook, "i_{k-1}"'] & C_{k-1}(X) \arrow[r, "\text{quotient}"] & C_k(X,A)
\end{tikzcd}$$

FIGURE 10.1.7.

where it is a simple fact of linear algebra that this diagram gives us a map $\partial_k^{(X,A)}$ for the quotient.

Thus we can form the *quotient chain complex*, which are groups defined dimension by dimension, to get $C_*(X, A)$. Then we can take $H_*(X, A)$.

REMARK 10.1.8. For A a subcomplex of X, these are nearly isomorphic to $H_*(X/A)$. In fact, $H_*(X, A) \cong H_*(X/A)$ for $* > 0$.

Now how are $H_*(A)$, $H_*(X)$, and $H_*(X, A)$ related? We would like to say that $H_*(X, A) = H_*(X)/H_*(A)$ but as we saw it is not that simple.

EXAMPLE 10.1.9. Let us do $H_*(D^{n+1}, S^n)$. Recall that $S^n = e^0 \cup e^n$, and $D^{n+1} = e^0 \cup e^n \cup e^{n+1}$. Then

$$C_*(D^{n+1}, S^n) = \begin{cases} \mathbb{Z} & * = n+1 \\ 0 & * \neq n+1 \end{cases},$$

where in dimension $(n+1)$ the $\mathbb{Z}$ is generated by e^{n+1}.

So the relative homology is similar:

$$H_*(D^{n+1}, S^n) = \begin{cases} \mathbb{Z} & * = n+1 \\ 0 & * \neq n+1 \end{cases}.$$

If we took $D^{n+1}/S^n = S^{n+1}$ the homology differs only in dimension 0.

Now in order to compare these groups we need some terminology from Linear Algebra.

10.2. Exact Sequences

We have already talked about chain complexes $C_k \to C_{k-1} \to \cdots$, which meant composites of two maps in a row are 0. Recall the idea of an *exact sequence*, where *exactness* at k means for $\cdots \to C_{k+1} \xrightarrow{f} C_k \xrightarrow{g} C_{k-1} \to \cdots$, $\mathrm{Im}(f) = \mathrm{Ker}(g)$, then a sequence is exact if it is exact everywhere.

There are some equivalent ways of defining a sequence of exact sequences:

A sequence of abelian groups and homomorphisms

$$\ldots \to C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} C_0$$

is an exact sequence if $\operatorname{Ker} f_n = \operatorname{Im} f_{n+1}$. Equivalently, this a chain complex for which the homology $H_k = 0$: recall that in a chain complex

$$f_k \circ f_{k+1} = 0 \text{ for all } k \iff \operatorname{Im} f_{k+1} \subset \operatorname{Ker} f_k,$$

then further we want $\frac{\operatorname{Ker} f_k}{\operatorname{Im} f_{k+1}} = 0$ to get equality.

Example 10.2.1. Suppose we have an exact sequence $0 \to A \to 0$. So all the maps are 0. Exactness here means that $\operatorname{Im}(0 \to A) = \operatorname{Ker}(A \to 0)$. But $\operatorname{Im}(0 \to A) = 0$, and $\operatorname{Ker}(A \to 0) = A$. So saying that this is exact is saying that $A = 0$.

This is a key thing to see about an exact sequence: an element in the sequence is 0 if its neighbors are 0.

Example 10.2.2. Suppose we have two nonzero terms, $0 \to A \to B \to 0$. Then exact means that $0 = \operatorname{Im}(0 \to A) = \operatorname{Ker}(A \to B) = 0$. So $A \hookrightarrow B$ is injective. By the same reasoning, $\operatorname{Im}(A \to B) = \operatorname{Ker}(B \to 0) = B$. So $A \to B$ is surjective. By combining these we conclude that $A \to B$ is an isomorphism.

The last elementary one is the one of length 3.

Example 10.2.3. Suppose we have $0 \to A \to B \to C \to 0$. This is called a *short exact sequence*. By the same reasoning as before, $A \to B$ is injective, and $B \to C$ is surjective. Exactness at B means $\operatorname{Im}(A \hookrightarrow B) = \operatorname{Ker}(B \to C)$, so $B/A = C$.

This is a very useful relationship that occurs often between groups. A simple example of this are split exact sequence.

Example 10.2.4. A *split (short) exact sequence* is where we have $B = A \oplus C$, then we have $0 \to A \to A \oplus C \to C \to 0$. This is called a split exact sequence since there is a map going back $C \to A \oplus C$.

In vector spaces, every exact sequence is split, since we can just lift the basis from C back to $B = A \oplus C$. Unfortunately for general groups the story is a bit more complicated.

Example 10.2.5. Suppose we have the exact sequence $0 \to \mathbb{Z}_2 \to ? \to \mathbb{Z}_2 \to 0$. Unfortunately, the solution is not unique. This could be $? = \mathbb{Z} \oplus \mathbb{Z}$, when it splits. But that is not the only possibility. Another possibility is where $? = \mathbb{Z}_4$, then $0 \to \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ is exact. This is not split.

So it is not true that the one in the middle is always completely uniquely determined. However, the size of the group is determined.

Let us look an example of an infinite group.

Example 10.2.6. For $0 \to \mathbb{Z} \to ? \to \mathbb{Z}_2 \to 0$ there are two possibilities. In the split case we could have $? = \mathbb{Z} \oplus \mathbb{Z}_2$, or we could have $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ so $? = \mathbb{Z}$ is possible.

This happens often in mathematics where we record the information for what we have control over but we cannot always pin down what the rest must be.

So what about a longer exact sequence? Well it gets more complicated, but we can say what is going on for many parts.

There is also a way of turning things into short exact sequences. We show it for a sequence of length 4.

Example 10.2.7. For $0 \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \to 0$, we know f is injective and h is surjective, but what about g? Well we can write down $0 \to A \to B \to \operatorname{Im} g \to 0$ and $0 \to \operatorname{Im} g \to B \to D \to 0$, so we get short exact sequences at the cost of introducing more groups into the picture.

So exact sequences are a good tool for writing down a lot of information.

Short exact sequences have already come up implicitly in the situation where we were looking at a pair of spaces (X, A), that is, we have $A \hookrightarrow X$ and we wanted to look at the relative homology groups and looked at

$$0 \to C_k(A) \to C_k(X) \to C_k(X, A) \to 0,$$

which is a short exact sequence, and took $C_k(X, A) = C_k(X)/C_k(A)$. But we could have just said that we have a short exact sequence.

Now we did not just have one exact sequences, we had one at every dimension, and maps ∂_k^A, ∂_k^X, and $\partial_k^{(X,A)}$ from each level to the one below it. Well if we write this as a tableau as in Figure 10.2.8 the columns are chain complex used to compute the homology groups $H_*(A)$, $H_*(X)$, $H_*(X/A)$.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_k(A) \arrow[r] \arrow[d, "\partial_k^A"] & C_k(X) \arrow[r] \arrow[d, "\partial_k^X"] & C_k(X,A) \arrow[r] \arrow[d, "\partial_k^{(X,A)}"] & 0 \\
0 \arrow[r] & C_{k-1}(A) \arrow[r] \arrow[d] & C_{k-1}(X) \arrow[r] \arrow[d] & C_{k-1}(X,A) \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Figure 10.2.8.

The naive hope was that we get a simple relationship, so that we would have a short exact sequence $0 \to H_k(A) \to H_k(X) \to H_k(X/A) \to 0$, but as we saw that this is not true since some of the holes in A can get filled in in X. This is where we got stuck.

Denote by a *short exact sequence of chain complexes*

$$0 \to C_*(A) \to C_*(X) \to C_*(X/A) \to 0$$

such a tableau where each row is a short exact sequence.

THEOREM 10.2.9 (Long exact sequence of a pair (X, A)). *There is a long exact sequence*

$$\ldots \to H_k(A) \to H_k(X) \to H_k(X, A) \xrightarrow{\partial}$$
$$H_{k-1}(A) \to H_{k-1}(X) \to H_{k-1}(X, A) \xrightarrow{\partial} \ldots$$

The proof is purely algebraic: a short exact sequence of chain complexes yields a long exact sequence of their homology groups.

PROOF. The construction of $\partial : H_k(X, A) \to H_{k-1}(A)$ is as follows: take $u \in C_k(X, A)$ with $\partial_k^{(X,A)}(u) = 0$.

Now $C_k(X) \to C_k(X, A)$ is surjective, so we can pick a lift $u \rightsquigarrow u' \in C_k(X)$. Now consider $\partial_k^X(u')$.

$$\begin{tikzcd}
 & u' \arrow[r, squiggly] \arrow[d, squiggly] & u \arrow[d, squiggly] \\
v \arrow[r, squiggly] & \partial_k^X(u') \arrow[r, squiggly] & 0
\end{tikzcd}$$

FIGURE 10.2.10.

Because the diagram commutes, $\partial_k^X(u') \rightsquigarrow 0$. But the short exact sequence says that $\partial_k^X(u')$ comes from $v \in C_{k-1}(A)$.

But why does v represent a homology class? For that we need $v \rightsquigarrow 0$ at the next level.

But $\partial_k^X(u') \rightsquigarrow 0$ going down the column since we have a chain complex, and then from exactness we have an injective map taking $\partial_{k-1}^A(v) \rightsquigarrow \partial_{k-1}^X \circ \partial_k^X(u') = 0$ going across.

$$\begin{tikzcd}
 & u' \arrow[r, squiggly] \arrow[d, squiggly] & u \arrow[d, squiggly] \\
v \arrow[r, squiggly] \arrow[d, squiggly] & \partial_k^X(u') \arrow[r, squiggly] \arrow[d, squiggly] & 0 \\
0 = \partial_{k-1}^A(v) \arrow[r, squiggly] & 0 &
\end{tikzcd}$$

FIGURE 10.2.11.

So $v \rightsquigarrow 0$.

The problem is that we have indeterminacy from the lift going from $C_k(A)$ by exactness of the row. But these do not matter in $H_{k-1}(A)$. So by the time we pass to homology, the choice is irrelevant, as it is in the denominator. So we have this sequence of maps.

To check that this is exact is a long painstaking process. We will just show one: take $H_k(A) \to H_k(X) \to H_k(X, A)$: well the composite is clearly 0, even for

chain complexes. So clearly on the homology, which is a quotient, the composite is 0 as well. So $\mathrm{Im}(H_k(A) \to H_k(X)) \subset \mathrm{Ker}(H_k(X) \to H_k(X, A))$.

Now suppose we have

$$[\alpha] \in \mathrm{Ker}(H_k(X) \to H_k(X, A)).$$

Well

$$C_k(X) \ni \alpha \rightsquigarrow \alpha' \in C_k(X, A),$$

which is 0 in $H_k(X/A)$. Now α' comes from $\beta' \in C_k(X, A)$, then we have

$$C_{k+1}(X) \ni \beta' \rightsquigarrow \beta \in C_{k+1}(X, A)$$

via an onto map, then we have $\partial^X_{k+1}(\beta')$.

Look at $\alpha - \partial^X_{k-1}(\beta')$, which goes to $\alpha - \alpha = 0$, so is an image of some $\gamma \in C_k(X)$. We claim that $\partial^A_k(\gamma) = 0$. This follows just as before: $(\alpha - \partial^X_{k-1}(\beta')) \rightsquigarrow 0$.

Lastly, we claim that $[\gamma] \in H_k(A)$ goes to $[\alpha] \in H_k(X)$. This is because $\gamma \rightsquigarrow (\alpha - \partial^X_k(\beta'))$, which differs from α by a boundary element, which represents 0 in homology. So we are done. □

This can be generalized to general short exact sequences of chain complexes $0 \to A_* \to B_* \to C_* \to 0$. So in fact the long exact sequence can be seen as a corollary of the more general long exact sequence of a short exact sequence of chain complexes.

Such long exact sequences with repetition every third term appears all over mathematics.

Example 10.2.12. Previously we computed with integer coefficients

$$\begin{aligned} H_*(S^n) &= \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{otherwise} \end{cases} \\ H_*(D^{n+1}) &= \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases} \\ H_*(D^{n+1}, S^n) &= \begin{cases} \mathbb{Z} & * = n+1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Then we have

$$\begin{aligned} 0 \to H_{n+1}(D^{n+1}, S^n) \to H_n(S^n) \to H_n(D^{n+1}) \to \ldots \to \\ 0 \to \ldots \to 0 \to H_0(S^n) \to H_0(D^{n+1}) \to 0 \to 0 \end{aligned}$$

Well we get $0 \to \mathbb{Z} = H_{n+1}(D^{n+1}, S^n) \xrightarrow{\cong} H_n(S^n) = \mathbb{Z} \to 0 \to 0$ from the long exact sequence.

There are also long exact sequences for homotopy groups and fibrations.

10.3. Relative Homology

Given a pair (X, A) and (Y, V), a *map of pairs* $(X, A) \xrightarrow{f} (Y, B)$ is a map $f : X \to Y$ with $f(A) \subset B$. It is easy to see that this induces a homomorphism $H_k(X, A) \xrightarrow{f_*} H_k(Y, B)$.

A cancellation property of homology called *excision* is as follows: we have $H_k(X, X) = 0$, but then we can generalize to the following:

THEOREM 10.3.1. *Suppose* $U \subset \mathrm{Int}(A) \subset X$, *then* $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$. *Then* $H_k(X \setminus U, A \setminus U) \to H_k(X, A)$ *is an isomorphism.*

We sketch a proof for the case of cell complexes.

PROOF SKETCH. Suppose U is the interior of a subcell complex in the interior of A. Then form $C_k(X, A) = C_k(X)/C_k(A) \cong C_k(X \setminus U)/C_k(A \setminus U)$. So we get isomorphism in the homology groups as well. □

EXAMPLE 10.3.2. Suppose we take the suspension ΣX of a space X, made of two cones $C_+(X)$ and $C_-(X)$.

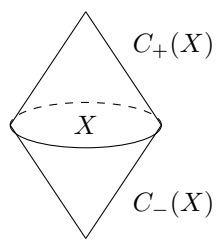

FIGURE 10.3.3.

Then $H_k(\Sigma X, C_+(X)) = H_k(C_-(X), X)$.

This allows us to cut things down by removing the same thing from the space and the subspace, provided what we are removing is in the interior.

We can do relative homotopy, but in that case we do not get excision.

The reason this is difficult to prove for homology in general is because in geometry, excision depends on what is called *transversality*. Well suppose we have a simplex in U. Removing it does nothing since it removes it from both A and X. The messy part is that the simplex might cut through the edge between U and A. So the key is to break the simplex up into things that do not cross the edge. This is hard to justify.

CHAPTER 11

A Discussion of the Axiomatic View of Homology

11.1. Axioms of Homology

We now build Homology up from an axiomatic point of view. In fact, we have already seen most of these. These axioms are known as the Eilenberg-Steenrod axioms of Homology.

For every pair (X, A), $A \subset X$, there are *natural* groups which are called $H_n(X, A)$, $n \in \mathbb{Z}$. Some conventional notation: $H_n(X, \emptyset)$ is called $H_n(X)$.

Naturality (or functoriality) here means that given pairs $(X, A) \xrightarrow{f} (Y, B)$ (that is $f : X \to Y$ with $f(A) \subset B$), there is an induced homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$ satisying $(\mathrm{Id}_{(X,A)})_* = \mathrm{Id}_{H_n(X,A)}$ and $(g \circ f)_* = g_* \circ f_*$.

Axiom 11.1.1 (Natural Long Exact Sequence). *The* first axiom *is that there is a natural long exact sequence of* (X, A)*:*

$$\dots H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial}$$
$$H_{n-1}(A) \to H_{n-1}(X) \to H_{n-1}(X, A) \xrightarrow{\partial} \dots$$

So if we know some of these groups, we can determine, up to a certain amount of ambiguity, what the other groups are.

Here the naturality (or functoriality) means furthermore that given a map of pairs $(X, A) \xrightarrow{f} (Y, B)$, we have the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(A) \arrow[r] \arrow[d, "(f|_A)_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X, A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r, "\partial"] \arrow[d, "(f|_A)_*"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y, B) \arrow[r] & H_{n-1}(B) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Figure 11.1.2.

Axiom 11.1.3 (Homotopy Axiom). *The* second axiom *is the homotopy axiom.*

A minimal form of this is as follows: Say we have $X \times I$*, and then* $X \overset{i}{\hookrightarrow} X \times 0$ *and* $X \overset{j}{\hookrightarrow} X \times 1$*, so* $i(u) = (u, 0)$ *and* $j(u) = (u, 1)$*. Then*

$$i_* = j_* : H_n(X) \to H_n(X \times I).$$

We actually need this for pairs (X, A)*, that is for* $(X, A) \times I = (X \times I, A \times I)$*.*

The stronger form is as follows: suppose we have $f, g : X \times Y$*, then*

$$f \underset{h}{\sim} g \implies f_* = g_* : H_n(X) \to H_n(Y).$$

In fact, these two forms are equivalent.

That the stronger form implies the minimal form is obvious; it is not so obvious to see that the minimal one implies the stronger one. But we just need to see that $f \underset{h}{\sim} g$ means that we have a map $H : X \times I \to Y$ with $H(\cdot, 0) = g(\cdot)$ and $H(\cdot, 1) = f(\cdot)$. So $g = H \circ i$ and $f = H \circ j$, then $g_* = H_* \circ i_*$ and $f_* = H_* \circ j_*$. But since $i_* = j_*$ we get $f_* = g_*$.

So a homotopy of pairs is just $(X, A) \times I \xrightarrow{H} (Y, B)$ is just $H : X \times I \to Y$ with $H(A \times I) \subset B$.

Corollary 11.1.4. *Homotopic spaces (or pairs) have isomorphic homologies.*

Proof. Given X, Y with $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$, $g \circ f \underset{h}{\sim} \mathrm{Id}_X$ and $f \circ g \underset{h}{\sim} \mathrm{Id}_Y$ means that $g_* \circ f_* = \mathrm{Id}_{H_n(X)}$ and $f_* \circ g_* = \mathrm{Id}_{H_n(Y)}$. □

Example 11.1.5. Note that $H_*(S^n \setminus k \text{ points}) \underset{h}{\sim} H_*(\bigvee_{k-1} S^{n-1})$. Then since $\bigvee_{k-1} S^{n-1} = e^0 \cup e^{n-1} \cup \ldots \cup e^{n-1}$, the homology groups are easy to compute.

Axiom 11.1.6 (Excision Axiom). *The* third axiom *is the excision axiom: given* $U \subset \overline{U} \subset \mathrm{Int}(A) \subset X$ *with* $(X \setminus U, A \setminus U) \overset{i}{\hookrightarrow} (X, A)$*, then*

$$H_n(X \setminus U, A \setminus U) \underset{\cong}{\overset{i_*}{\to}} H_n(X, A).$$

Now everything we have said does not prevent us from doing something vacuous. The final axiom fixes this.

Axiom 11.1.7 (Dimension Axiom). *The* fourth axiom*, the dimension axiom, says that for coefficients in a ring* R *(eg.* $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_p, \mathbb{Z}_2$*)*

$$H_*(\textit{point}) = \begin{cases} R & \textit{in dimension } 0 \\ 0 & \textit{otherwise} \end{cases}$$

When we derive consequences, we prefer not to use the dimension axiom, since it is of a different character from all of the rest. If we can avoid the dimension axiom, we can apply the results to more general (extraordinary) Homology theories that capture the first three axioms but not the fourth.

These axioms changed the field beautifully when they came out in the 1950s. Now there are various "homologies" that are analogous but not proper homology theories in the sense of these axioms.

There are two versions of a uniqueness theorem for homology.

Theorem 11.1.8 (Uniqueness Theorem). *Any theory satisfying the four axioms are isomorphic to the usual Homology theory with coefficients with* R*,* $H_*(\cdot; R)$*.*

We will compute from the axioms what the homology of a suspension $H(\Sigma X)$ is. Recall that $\Sigma S^k = S^{k+1}$, and if we suspend k times, $\Sigma^j S^k = S^{k+j}$.

We compute the homology of the suspension. So roughly $H_*(\Sigma X) \cong H_{*-1}(X)$. Of course this cannot be exactly right, since $H_0(\Sigma X) = H_0(X) = R$. So for $X \neq \emptyset$ we have a map $X \xrightarrow{f} \{\text{point}\}$, then we have $H_*(X) \xrightarrow{f_*} H_*(\text{point})$.

Well we have

$$H_*(X) \cong \mathrm{Ker}(f_*) \oplus H_*(\text{point}).$$

The way to see this is that if point $\overset{i}{\hookrightarrow} X \overset{f}{\to}$ point then

$$f_* \circ i_* = (f \circ i)_* = (\mathrm{Id}_{\text{point}})* = \mathrm{Id}_{H_n(\text{point})}.$$

$\mathrm{Ker}(f_*)$ is called the *reduced homology* $\overline{H}_*(X)$. So

$$\overline{H}_*(S^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise} \end{cases} \qquad \overline{H}_*(S^0) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}.$$

In relative homology $\overline{H}_*(X, A) = H_*(X, A)$ since we remove the same thing from both.

THEOREM 11.1.9. $\overline{H}_*(\Sigma X) = \overline{H}_{*-1}(X)$.

Then to get back the regular Homology we can just add back in the homology of a point everywhere.

PROOF. We do this in three steps.

The first step is $\overline{H}_*(\Sigma X) = H_*(\Sigma X, C_+X)$, where C_+X is the upper cone of ΣX.

Consider the exact sequence of the pair $(\Sigma X, C_+X)$:

$$\overline{H}_*(C_+X) \to \overline{H}_*(\Sigma X) \to H_*(\Sigma X, C_+X) \overset{\partial}{\to} \overline{H}_*(C_+X) \to \dots$$

But $C_+X \underset{h}{\sim} \{\text{point}\}$, $\overline{H}_*(\{\text{point}\}) = 0$, so in fact we have

$$0 \to \overline{H}_*(\Sigma X) \overset{\cong}{\to} H_*(\Sigma X, C_+X) \overset{\partial}{\to} 0 \to \dots$$

The next step uses excision: $(C_+X, X) \hookrightarrow (\Sigma X, C_+X)$ implies via excision that $H_*(C_-X, X) \overset{\cong}{\to} H_*(\Sigma X, C_+X)$.

For the final step, consider the pair (C_-X, X):

$$\dots \to \overline{H}_*(C_-X) \to H_*(C_-X, X) \overset{\partial}{\to} \overline{H}_{*-1}(X) \to \overline{H}_{*-1}(C_-X) \to \dots$$

But again, $C_-X \underset{h}{\sim} \{\text{point}\}$, so $H_*(C_-X, X) \cong \overline{H}_{*-1}(X)$.

Then putting this all together we get $\overline{H}_*(\Sigma X) \cong \overline{H}_{*-1}(X)$. □

Notice that we did not need to use the dimension axiom here since we removed the homology of a point.

EXAMPLE 11.1.10. What is $H_*(S^n)$? Well $H_*(S^n) = H_*(\text{point}) \oplus \overline{H}_*(S^n)$ where

$$\overline{H}_*(S^n) \cong \overline{H}_{*-1}(S^{n-1}) \cong \dots \cong \overline{H}_{*-n}(S^0) \cong \overline{H}_{*-n}(\text{point})$$

So $H_*(S^n) = H_*(\text{point}) \oplus H_{*-n}(\text{point})$.

So we begin to see why there can be a uniquness theorem, since we can just build up cell-complexes using these spheres.

11.2. H_0 and H_1

Let us look a bit more at H_0 and H_1.

PROPOSITION 11.2.1. $H_0(X; R) = R^{\#\textit{path connected components of } X}$.

Proof. Let us look at the singular chain complex: $C_1(X;R) \xrightarrow{\partial} C_0(X;R)$. So what is $C_0(X;R)$? Well 0-simplices are just points so $C_0(X;R) = \bigoplus_{\text{points in } X} R$. 1-simplices are just paths, so $C_1(X;R) = \bigoplus_{\text{paths in } X} R$. Now what does the boundary map ∂ do? Well if we have a path ω taking $p \rightsquigarrow q$, then $\partial\omega = q - p$. So $H_0(X;R) = (\bigoplus_{\text{points in } X} R)/\{(q-p) \mid \exists \text{ path } p \rightsquigarrow q\}$. So we send $q - p = 0$ so $q = p$, that is, points are equal to each other if there are paths between them, so we end up with just one copy of R for each path-connected component of X. $\square$

Proposition 11.2.2. *If X is path connected then $H_1(X,\mathbb{Z}) \equiv \pi_1(X,x)/[\cdot,\cdot]$ (where $[\cdot,\cdot]$ is the commutator), that is, π_1 made abelian.*

The relation between higher homology and homotopy groups are much more complicated.

Proof. The plan is to create a homomorphism $\pi_1(X,x)/[\cdot,\cdot] \xrightarrow{h} H_1(X;\mathbb{Z})$ and see that the kernel is exactly $[\cdot,\cdot]$. This map is called the Hurewicz map.

So take $[\alpha] \in \pi_1(X,x)$. Now $\alpha : \Delta^1 = I \to X$ with $\alpha(0) = \alpha(1)$, so $\alpha \in C_1(X)$, with boundary 0. So it represents an element $[\alpha] \in H_1(X;\mathbb{Z})$.

We need to check that this map is well-defined, and that it is a homomorphism. Then we need to check its kernel and that it is surjective.

So for well-definedness, suppose we picked another loop $\alpha' \underset{h}{\sim} \alpha$.

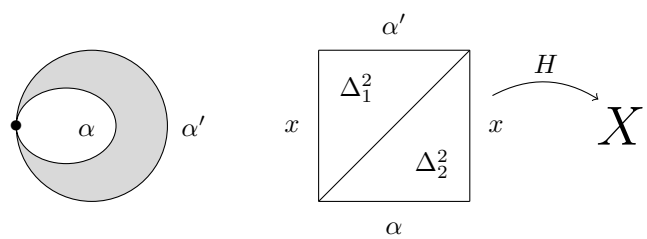

Figure 11.2.3.

Well the homology is a map $I \times I \xrightarrow{H} X$, where $(\cdot,0) = \alpha$, $(\cdot,1) = \alpha'$, and then $(0,\cdot) = (1,\cdot) = x$. We can cut this up into two 2-simplices Δ_1^2 and Δ_2^2 by cutting from $(0,0)$ to $(1,1)$, and we have a trivial 1-simplex Δ^1 at x. Then

$$\partial(\Delta_1^2 + \Delta_2^2) = (\alpha) + \Delta^1 + \text{diagonal} + (-\alpha') - \Delta^1 - \text{diagonal} = \alpha - \alpha'$$

So $[\alpha] = [\alpha']$ in $H_1(X,\mathbb{Z})$.

Next, why does $[\alpha \cdot \beta] = [\alpha] + [\beta]$ in $H_1(X,\mathbb{Z})$? Well we can get a triangle from α, β, $\alpha \cdot \beta$:

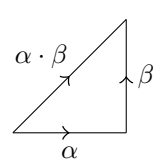

Figure 11.2.4.

Then $\partial\Delta^2 = \alpha + \beta - (\alpha \cdot \beta)$, but the boundary of this triangle is degenerate so $[\alpha] + [\beta] = [\alpha \cdot \beta]$ in $H_1(X;\mathbb{Z})$.

Now we can say that since $H_1(X;\mathbb{Z})$ is abelian, the map $\pi_1(X) \to H_1(X;\mathbb{Z})$ factors through $\pi_1(X)/[\cdot,\cdot]$ for free:

$$\begin{tikzcd} \pi_1(X) \arrow[rr] \arrow[dr] & & H_1(X) \\ & \pi_1(X)/[\cdot,\cdot] \arrow[ur] & \end{tikzcd}$$

FIGURE 11.2.5.

So to get an isomorphism we want a homomorphism $H_1(X;\mathbb{Z}) \to \pi_1(X)/[\cdot,\cdot]$.

Remember that $H_1(X;\mathbb{Z})$ is a subquotient of $C_1(X)$. So we will define a homomorphism $C_1(X) \to \pi_1(X)/[\cdot,\cdot]$, and from this we will get a homomorphism on the homology on its subquotient $H_1(X;\mathbb{Z})$.

The problem is that a typical chain α is far from being a loop. But we can just connect everything to the basepoint: for each $p \in X$, pick a path γ_p taking $p \rightsquigarrow x$ (take γ_x be the trivial path). Now if $\alpha(0) = p$ and $\alpha(1) = q$ then we can take $\alpha \overset{\Phi}{\rightsquigarrow} \gamma_p^{-1}\alpha\gamma_q \in \pi_1(X,x)/[\cdot,\cdot]$.

We need to check that this is a homomorphism, and that $\Phi(\partial C_2(X)) = 0$ in $\pi_1(X)/[\cdot,\cdot]$. Together these give a homomorphism

$$H(X;\mathbb{Z}) \subset C_1(X)/\partial C_2(X) \to \pi_1(X)/[\cdot,\cdot].$$

But the homomorphism of Φ is obvious since $C_1(X)$ was a free abelian group defined on a basis of $\bigoplus \mathbb{Z}$.

Now take a 2-simplex with endpoints p, q, r. In order to turn this into a loop, we take the loop $x \rightsquigarrow r \rightsquigarrow p \rightsquigarrow x \rightsquigarrow p \rightsquigarrow q \rightsquigarrow x \rightsquigarrow q \rightsquigarrow r \rightsquigarrow x$.

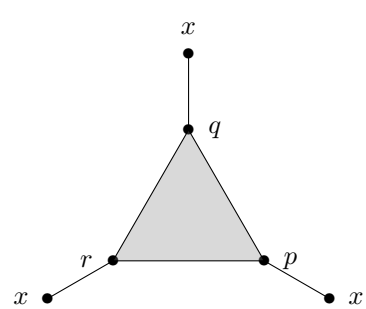

FIGURE 11.2.6.

Obviously this can be filled in, so the boundary maps to 0.

So we have maps $\pi_1(X)/[\cdot,\cdot] \to H_1(X;\mathbb{Z})$ and back. It is easy to check that the composites are identities. $\square$

EXERCISE 11.2.7. Check the details on these composites.

In general there is a homomorphism $\pi_n(X) \to H_n(X)$, called Hurewicz homomorphism, but unfortunately it is not in general injective or surjective.

In words, it is just mapping spherical holes to all holes. But not every hole is spherical, and in homotopy we can only fill in holes by disks but homology has no such standard. There is one important case when it is isomorphic, and a slightly less important case where it is surjective.

Roughly if there is a space with no holes below dimension n for $n > 1$, then in the first dimension with non-trivial holes, the homotopy and homology groups are the same, since in the first such dimension there is no way to make non-spherical holes. For $n = 1$ we cannot say this since π_1 is not necessarily abelian.

Exercise 11.2.8. Compare π_1(Klein bottle) to H_1(Klein bottle).

11.3. The Homotopy Axiom

We have discussed the other axioms in detail as part of the various forms of Homology; let us now talk about this one.

So why is Homology a Homotopy invariant?

Well we just need to consider a key special case where we have a cylinder $X \times I$ with $X \overset{i}{\hookrightarrow} X \times 0$, $i(u) = (u, 0)$ and $X \overset{j}{\hookrightarrow} X \times I$, $j(u) = (u, 1)$.

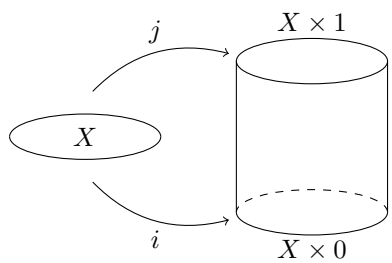

Figure 11.3.1.

What we need is for $i_* = j_* : H_k(X) \to H_k(X \times I)$.

Say for the cellular chain complex, for every cell u in X, $s(u)$ will be the cell $u \times I$ in $X \times I$. Correspondingly, we produce $s : C_k(X) \to C_{k+1}(X \times I)$. Now what is the boundary of this $s(u)$? Well this is difficult because when we thicken it up, we get a top, a bottom, and sides. So $\partial s(u) = j(u) - i(u) \pm s(\partial u)$. That is, $(\partial s \pm s\partial)(u) = j(u) - i(u)$.

Claim 11.3.2. *The left hand side $(\partial s \pm s\partial)(u)$ will vanish in the Homology.*

Proof. In a Homology $\partial u = 0$ since it is on the boundary, so $(s\partial)(u) = 0$. The other term is $(\partial s)(u) = 0$ anyway, since it is on the boundary as well. $\square$

So at the level of Homology, in fact $i_* = j_*$.

To make this more precise, say in Singular Homology, we introduce the notion of a *(algebraic) chain homotopy* between maps of chain complexes.

We talked about maps of chain complexes, where for chains $A_k \overset{\partial_k^A}{\to} A_{k-1} \to \cdots$ and $B_k \overset{\partial_k^B}{\to} B_{k-1} \to \cdots$ then (f_k) is a map of chain complexes so that

$$\partial_k^B \circ f_k = f_{k-1} \circ \partial_k^A.$$

We now need to do something more. In Topology we talk about Homotopies, so let us talk about the algebraist's version of Homotopy.

Well if we have a map of chains, what is the homotopy?

Definition 11.3.3. Suppose we are given chain complexes A_*, B_* and maps between them $A_* \overset{(f_*)}{\to} B_*$ and $A_* \overset{(g_*)}{\to} B_*$. A *chain homotopy* of f_* and g_* is a family of maps $A_k \overset{(s_k)}{\to} B_{k+1}$ satisfying $f_k - g_k = \partial^B_{k+1} s_k \pm s_{k-1}\partial^A_k$.

$$\begin{tikzcd}
{} \arrow[d] & & {} \arrow[d] \\
A_{k+1} \arrow[r, shift left, "f_{k+1}"] \arrow[r, shift right, "g_{k+1}"'] \arrow[d] & B_{k+1} \arrow[d] \\
A_k \arrow[r, shift left, "f_k"] \arrow[r, shift right, "g_k"'] \arrow[d] \arrow[ur, dashed, "s_k"] & B_k \arrow[d] \\
A_{k-1} \arrow[r, shift left, "f_{k-1}"] \arrow[r, shift right, "g_{k-1}"'] \arrow[d] \arrow[ur, dashed, "s_{k-1}"] & B_{k-1} \arrow[d] \\
{} & {}
\end{tikzcd}$$

Figure 11.3.4.

Proposition 11.3.5. *If there is such a chain homotopy (f_*) to (g_*), then the induced maps on Homology are the same: $H_*(A_*) \overset{f_*=g_*}{\to} H_*(B_*)$.*

Proof. On $\mathrm{Ker}(\partial^A_k)$, we have $s_{k-1}(\partial^A_k) = 0$, and $\partial^B_{k+1} s_k = 0$ in Homology. So $f_*[u] = g_*[u]$ for $u \in A$. $\square$

So we'll see that a Homotopy in Topology gives a Chain Homotopy of chain complexes, which means the same maps in Homology. The second part we have seen, and the first part we sketched for the Cellular chain complex. We will also do it now for Singular chain complexes.

So let us go back to the cylinder $X \times I$ with $X \overset{i}{\hookrightarrow} X \times 0$, $i(u) = (u, 0)$ and $X \overset{j}{\hookrightarrow} X \times I$, $j(u) = (u, 1)$. So say we have a singular simplex $\Delta^k \overset{\alpha}{\to} X$. We would like to construct $s(X)$, which is a sum of singular simplices in $X \times I$. Recall that this corresponds geometrically to $\alpha \times I$. The problem is that $\Delta^k \times I$ is a prism, not a simplex.

So what we will need to do is decompose $\Delta^k \times I$ into a sum of simplices and work with those.

Now for example, $\Delta^1 \times I$ is just a square, so it easy to break it up into simplices. For general $\Delta^k \times I$, well Δ^k has vertices $v_0, v_1, \ldots, v_k$, then $\Delta^k \times I$ has vertices $v_0 \times 0, v_1 \times 0, \ldots, v_k \times 0, v_0 \times 1, v_1 \times 1, \ldots, v_k \times 1$.

The idea is to go around the bottom for a while and then jump to the top, or jump up and then stay up at the top. In $\Delta^1 \times I$ we can do $v_0 \times 0, v_1 \times 0, v_1 \times 1$, and $v_0 \times 0, v_0 \times 1, v_1 \times 1$.

So we take $v_0 \times 0, v_1 \times 0, \ldots, v_j \times 0, v_j \times 1, v_{j+1} \times 1 \ldots, v_k \times 0$. So these are $k+2$ simplices, and so $\Delta^k \times I$ is a union of $k+1$ Δ^{k+2}'s.

For Δ a simplex, let $s(\Delta^k) = \sum_{j=0}^{k+1} \Delta_j^{k+1}$. Then we can check that

$$j(\Delta^k) - i(\Delta^k) = \partial \circ s_k \pm s_{k-1} \circ \partial$$

so we get a chain homotopy s.

Well let us go back to $\Delta^1 \times I$. We have

$$s(\langle v_0, v_1 \rangle) = \langle v_0 \times 0, v_1 \times 0, v_1 \times 1 \rangle + \langle v_0 \times 0, v_0 \times 1, v_1 \times 1 \rangle$$

so

$$\begin{aligned} \partial s + s\partial(\langle v_0, v_1 \rangle) &= \partial(\langle v_0 \times 0, v_1 \times 0, v_1 \times 1 \rangle + \langle v_0 \times 0, v_0 \times 1, v_1 \times 1 \rangle) \\ &\quad + s(v_1 - v_0) \\ &= \langle v_0 \times 0, v_1 \times 0 \rangle + \langle v_1 \times 0, v_1 \times 1 \rangle + \langle v_0 \times 0, v_1 \times 1 \rangle \\ &\quad - (\langle v_0 \times 0, v_0 \times 1 \rangle + \langle v_0 \times 1, v_1 \times 1 \rangle + \langle v_0 \times 0, v_1 \times 1 \rangle) \\ &\quad + v_1 - v_0 \end{aligned}$$

which vanishes.

This generalizes just as it did before.

CHAPTER 12

Miscellaneous Topics

12.1. Application: Lefschetz Fixed Point Theorem

The Lefschetz Fixed Point Theorem is a generalization of the Brouwer Fixed Point Theorem.

So let X be a finite cell complex, so in particular X is compact. Consider a function $f : X \to X$. We want to know if what the fixed points are.

EXAMPLE 12.1.1. For $\mathrm{Id}_{S^1} : S^1 \to S^1$ every point is fixed, but if we rotate it, no points are fixed, but these maps are homotopic.

So not all homotopy classes have a fixed point. But we will see that for most (in some sense) homotopy classes, all maps in the class have a fixed point. Obviously this is not the same point, but there is a fixed point.

Note that this does not work for non-compact spaces; in fact for those spaces it is kind of hopeless. To do so we need much more structure.

THEOREM 12.1.2 ((Hopf-)Lefschetz Fixed Point). *For $f : X \to X$, consider the induced map $f_k : H_k(X) \to H_k(X)$. Fixing a basis for $H_k(X)$, this is a matrix, so we can consider the trace* $\mathrm{Tr}(f_k)$. *Then if the* Lefschetz number

$$L(f) = \sum_k (-1)^k \,\mathrm{Tr}(f_k)$$

is non-zero, then f has a fixed point.

Note that the sum only depends on the homotopy class.

EXAMPLE 12.1.3. Take $X = D^n$. As we saw $H_*(D^n) = 0$ for $* \neq 0$ and $H_0(D^n) = \mathbb{Z}$. So $f_0 : \mathbb{Z}^{\#\text{components}} \to \mathbb{Z}^{\#\text{components}}$, that is $f_0 = (1)_{1\times 1}$. So $L(f) = (-1)^0 \times 1 = 1 \neq 0$. So f has a fixed point. More general we can take X to be anything contractible.

EXAMPLE 12.1.4. Take $X = \mathbb{R}\mathrm{P}^2$. $H_*(\mathbb{R}\mathrm{P}^2; \mathbb{Q}) = 0$ if $* > 0$ and $H_0(\mathbb{R}\mathrm{P}^2; \mathbb{Q})$, so this behaves like a point using rational coefficients. So any function $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ has a fixed point.

EXAMPLE 12.1.5. Take $X = S^n$ (Hadamard's Theorem). For $f : S^n \to S^n$, $f_0 = \mathbb{Z}$ and $f_n : H_n(S^n) \to H_n(S^n)$ is $f_n = \deg(f)$. So $L_f = 1 + (-1)^n \deg(f)$. So if $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.

Note that we really do this exception, since the antipodal map $S^n \xrightarrow{\alpha} S^n$ has no fixed points. It is not hard to see that the degree is $\deg(\alpha) = (-1)^{n+1}$.

Hadamard proved this analytically, not topologically.

EXAMPLE 12.1.6. Suppose $f \underset{h}{\sim} \mathrm{Id}$. Then using real coefficients, the Lefschetz number is

$$L(f) = \sum_k (-1)^k \operatorname{Tr}(\mathrm{Id}_{H_k(X)}) = \sum_k (-1)^k \beta_k(X)$$

where $\beta_k(X)$ is the k-th *Betti number*, which is the dimension of $H_k(X;\mathbb{R})$.

For every space, $\chi(X) = \sum_k (-1)^k \beta_k(X)$ is the Euler characteristic of X (Algebraic Geometers use e instead of χ). Another way to write the Euler characteristic is $\sum_k (-1)^k (\#k - \text{cells of } X)$. Recall that for a graph on the $S^2 = \mathbb{R}^2 \cup \{\infty\}$, we have $v - e + f = 2$.

The proof that the two sums are equal will lead to the proof of the Lefschetz Fixed Point Thoerem.

So $\chi(X)$ is an important invariant of spaces based on the following theorem:

THEOREM 12.1.7. *Let X be a finite cell-complex. Then we have*

$$\sum (-1)^i c_i(X) = \sum (-1)^i \beta_i(X)$$

where $c_i(X)$ is the number of i-cells in the cell decomosition, and the Betti numbers $\beta_i(X) = \operatorname{Rank}(X;R)$, where R is a ring $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$.

In fact $R = \mathbb{Z}_2$ gives different Betti numbers, but the overall sum ends up being the same.

The theorem is proved by applying the following proposition to the cellular chain complex:

PROPOSITION 12.1.8. *Given a chain complex of finitely generated groups (vector spaces over a field) $C_i \to C_{i-1} \to \ldots \to C_0$, call it $\mathcal{C}$, then*

$$\sum (-1)^i \operatorname{Rank} C_i = \sum (-1)^i \operatorname{Rank} H_i(\mathcal{C}).$$

For example, in a short exact sequence (so $H_i = 0$) $0 \to A \to B \to C \to 0$, then

$$\operatorname{Rank} B = \operatorname{Rank} A + \operatorname{Rank} C \implies \operatorname{Rank} A - \operatorname{Rank} B + \operatorname{Rank} C = 0.$$

PROOF. What we will do is break the long chain complex into short sequences, then add them back up and see what we get.

Look at $C_i \to C_{i-1} \to C_{i-2}$; we have the cycles $Z_i = \operatorname{Ker}(C_i \to C_{i-1})$ and boundaries $B_{i-1} = \operatorname{Im}(C_i \to C_{i-1})$. Then $B_i \subseteq Z_i \subseteq C_i$ and $H_i = Z_i/B_i$.

So we have the short exact sequence

$$0 \to B_{i-1} \to Z_{i-1} \to H_{i-1} \to 0,$$

so

$$\operatorname{Rank} Z_{i-1} = \operatorname{Rank} H_{i-1} + \operatorname{Rank} B_{i-1}.$$

Furthermore, we have

$$0 \to Z_{i-1} \to C_{i-1} \to B_{i-2} \to 0,$$

so

$$\operatorname{Rank} C_{i-1} = \operatorname{Rank} Z_{i-1} + \operatorname{Rank} B_{i-2}.$$

Take the alternating sum

$$\sum (-1)^i \operatorname{Rank} C_i = \sum (-1)^i \operatorname{Rank} Z_i + \sum (-1)^i \operatorname{Rank} B_{i-1}.$$

But then plugging in for $\operatorname{Rank} Z_i$ we get

$$\sum(-1)^i \operatorname{Rank} C_i = \sum(-1)^i \operatorname{Rank} H_i + \sum(-1)^i \operatorname{Rank} B_i + \sum(-1)^i B_{i-1}$$

but the last two sums cancel, so when we are all finished we are left with

$$\sum(-1)^i \operatorname{Rank} C_i = \sum(-1)^i \operatorname{Rank} H_i \qquad \square$$

An amusing application is as follows: For n odd, S^n has a free action of $\mathbb{Z}_k$ for each k. One way to do this is S^{2m-1} as a unit sphere in $\mathbb{C}^m$, then one example of a free action is scalar multiplication by $\eta = e^{2\pi i/k}$, which is a k-th root of unity. S^{2m-1}/η is a lens space $L^{2m-1}(k)$ (there is a generalization where we use different roots of unity at different coordinates) and these have $\pi_1 = \mathbb{Z}_k$. When $k = 2$, then the quotient just gives $\mathbb{R}\mathrm{P}^2$.

Proposition 12.1.9. *For $k > 2$, there is no free action of $\mathbb{Z}_k$ on S^{2m}.*

Proof. To make the proof easier, we will assume that the quotient has a cell-complex structure. Now let $Y \to X$ be a finite cover of degree d. Then $\chi(Y) = d\chi(X)$, since we can use a cell decomposition of X to get a cell decomposition of Y with d cells in Y over each cell in X.

But this gives implications for divisibility. In particular, for a $\mathbb{Z}_k$ free action on S^{2m}, we get a degree k covering map $S^{2m} \to S^{2m}/\mathbb{Z}_k$, then $\chi(S^{2m}) = k\chi(S^{2m}/\mathbb{Z}_k)$, so $k|\chi(S^{2m}) = 1 + (-1)^{2m} = 2$, so k cannot be bigger than 2. $\square$

So back to the Lefschetz number, we can think of this as a generalization of this.

Proof Sketch of Theorem 12.1.2. Assume (for simplicity) that X has a cell decomposition. We can assume after subdividing the cells that f is approximated by (and in particular homotopic to) what is called a cellular map f' which maps each cell to a cell of at most the same dimension. So the induced map f'_* can be a map $f'_* : C_i(X) \to C_i(X)$.

Then by the same argument as for the Euler characteristic we will show that

$$\sum(-1)^i \operatorname{Tr}(C_i \overset{(f')_*}{\to} C_i) = \sum(-1)^i \operatorname{Tr}(H_i(X) \overset{(f')_*}{\to} H_i(X)) = L(f).$$

As an example on a short exact sequence $0 \to A \to B \to C \to 0$, then a map from this to itself is comprised of $\alpha : A \to A$, $\beta : B \to B$, $\gamma : C \to C$. But $\beta = \begin{pmatrix} \alpha & 0 \\ * & \gamma \end{pmatrix}$, so $\operatorname{Tr}\beta = \operatorname{Tr}\alpha + \operatorname{Tr}\gamma$.

Then $L(f) \neq 0$ means that for some dimension f'(cell) is inside that cell. So there is a point whose image, as it lies in the same cell, is not far away from the point.

Now keep subdividing the cells, and take the limit of such points. As we subdivide, we also can make f' approximate f better and better. $\square$

We could have also used simplices instead of cell complexes.

So most maps have fixed points.

12.2. The Mayer-Vietoris Theorem

This is the analogue of van Kampen's theorem but for $H_i(X)$. This is for $X = A \cup B$.

For simplicity, let X, A, B be cell complexes, with $A \cap B$ likewise. Well then $\#k$-cells of $X = \#k$-cells of $A + \#k$-cells of $B - \#k$-cells of $A \cap B$.

So we have the following inclusions:

$$\begin{tikzcd}
 & A \arrow[dr, hook, "i"] & \\
A \cap B \arrow[ur, hook, "k"] \arrow[dr, hook, "\ell"'] & & X \\
 & B \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

FIGURE 12.2.1.

Then we have the short exact sequence

$$0 \to C_*(A \cap B) \overset{k_* - \ell_*}{\to} C_*(A) \oplus C_*(B) \overset{i_* + j_*}{\to} C_*(X) \to 0.$$

The *Mayer-Vietoris sequence* now runs as follows:

$$\ldots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(X) \overset{\partial}{\to} H_{i-1}(A \cap B) \to \ldots$$

This can be derived from the Homology axioms without the dimension axiom. The trickiest part is to derive the boundary map ∂^{MV}, which is obtained as follows: suppose we have $H_i(X) \to H_i(X, A)$, then by excision we have $H_i(X, A) \equiv H_i(B, A \cap B)$. But we have a boundary map $H_i(B, A \cap B) \overset{\partial}{H}_{i-1}(A \cap B)$. It is a lot of work to show how this fits into a long exact sequence.

EXAMPLE 12.2.2. Take $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$. Then we can get

$$\ldots \to H_k(D^n_-) \oplus H_k(D^n_+) \to H_k(S^n) \overset{\partial}{\to} \\ H_{k-1}(S^{n-1}) \to H_{k-1}(D^n) \oplus H_{k-1}(D^n) \to \ldots$$

Then since the homologies of disks are 0 in most dimensions, we obtain that $H_k(S^n) \underset{\cong}{\overset{\partial}{\to}} H_{k-1}(S^{n-1})$ except in dimension 0.

Note that we can see $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, either by counting cells, or by the Mayer-Vietoris sequence.

12.3. Homology of a Product Space

The homology of a product space is tricky. The homotopy was very simple: $\pi_k(X \times Y) = \pi_k(X) \times \pi_k(Y)$. One of the significant differences between homotopy and homology is that in homology there is no formula like that, and there is a reason why.

PROPOSITION 12.3.1. $\chi(X \times Y) = \chi(X)\chi(Y)$.

PROOF. If we count cells, i-cell in X, j-cell in Y, then the product is $(i+j)$-cell in $X \times Y$. Then χ just counts the cells (with signs of parity of dimension). $\square$

But χ can also be computed from the Homology $H_*(\cdot)$. This suggests that $H_k(S^1 \times S^1) = \bigoplus_{i+j=k} H_i(S^1) \otimes H_j(S^1)$.

The Kunneth Theorem says that this is valid for coefficients in a field, and for coefficients in $\mathbb{Z}$ if at least one of X, Y has no torsion in its Homology. But there is a further torsion term when both X, Y have torsion with $\mathbb{Z}$ coefficients.

Now there are some nice maps from a product space $X \times Y$. There are the projection maps $X \times Y \xrightarrow{p_1} X$ and $X \times Y \xrightarrow{p_2} Y$. These accounted for the Homotopy groups but not for the Homology group because of the cross-talk. So what is left over when we take out X and take out Y?

The Smash product will get rid of X and Y in $X \times Y$.

Definition 12.3.2. The *smash product* is $X \wedge Y = (X \times Y)/(X \vee Y)$.

Example 12.3.3. $S^k \wedge S^\ell = S^{k+\ell}$, which we can see by taking cells: we have $(e^0 \cup e^k) \wedge (e^0 \cup e^\ell)/(e^0 \times (e^0 \cup e^\ell) \cup (e^0 \cup e^k) \times e^\ell)$, which leaves $e^0 \cup e^{k+\ell} = S^{k+\ell}$.

So we also have the map $X \times Y \xrightarrow{q} X \wedge Y$. These maps together detect all of the Homology $H_*(X \times Y)$.

Now we are tempted to say if they give the whole thing, why don't we put them together and decompose $X \times Y$ into them? Well the problem is that these maps have nothing to do with each other. So there is no way to directly compare $X \times Y$ with, for example $X \vee Y \vee (X \wedge Y)$. Now they have isomorphic Homology but there are no maps between them. In fact they are not the same Homotopically.

Now let us look at the case of a Torus.

Example 12.3.4. If we take $S^1 \times S^1$ this has the same Homology as the space $S^1 \vee S^1 \vee S^1 \wedge S^1 = S^1 \vee S^1 \vee S^2$. But we have no maps comparing them. However, when we have suspension, we can add maps together. So we can take these maps p_1, p_2, q and add them up, and then we can have a comparison going.

So when we take the suspension, we get $\Sigma(X \times Y) \xrightarrow{\Sigma p_1} \Sigma X$, $\Sigma(X \times Y) \xrightarrow{\Sigma p_2} \Sigma Y$, and $\Sigma(X \times Y) \xrightarrow{\Sigma 1} \Sigma X \wedge Y$. Now we can take these three maps and combine them into $\Sigma(X \times Y) \xrightarrow{f} \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ where $f = (\Sigma p_1) + (\Sigma p_2) + (\Sigma q)$.

It is not hard to check that this induces an isomorphism on the Homology $H_*(\cdot)$.

Theorem 12.3.5 (Whitehead). *If we have a map $V \xrightarrow{f} W$ of simply connected spaces, then the following are equivalent:*

(1) f is a homotopy equivalence.
(2) f_* is an isomorphism on $H_*(\cdot; \mathbb{Z})$.
(3) f_* is an isomorphism on $\pi_*(\cdot)$.

It would be tempting to say that if two spaces have the same Homology, then they are Homotopy equivalent. However, it depends on the existence of a geometric map. So if we do have a geometric map and it induces an isomorphism on Homology, then this is the criteria for Homotopy equivalence.

Example 12.3.6. If we take $S^p \times S^q$, $\Sigma(S^p \times S^q) \xrightarrow{h.e.} \Sigma S^p \vee \Sigma S^q \vee \Sigma(S^p \wedge S^q)$, but this says that $\Sigma(S^p \times S^q) \underset{h}{\sim} S^{p+1} \vee S^{q+1} \vee S^{p+q+1}$.

So for the specific case of $S^1 \times S^1$, the suspension of a torus is homotopy equivalent to $S^2 \vee S^2 \vee S^3$.

Note that while the Whitehead theorem fails for non-simply connected spaces. Homotopy will catch simply connectedness, but Homology is not good enough of an invariant. For example, π_1 could abelianize to 0 and we lose all of the information.

But there is a trick: for non-simply connected spaces, we pass to the universal cover:

THEOREM 12.3.7. *If we have $f : V \to W$ a map of connected spaces, then the following are equivalent:*

(1) f is a homotopy equivalence.

(2) f_* is an isomorphism on $\pi_1(\cdot)$, and then $\tilde{X} \xrightarrow{\hat{f}} \tilde{Y}$ induces an isomorphism $H_*(\tilde{X}) \xrightarrow{\hat{f}_*} H_*(\tilde{Y})$.

(3) f_* is an isomorphism on $\pi_*(\cdot)$.

So this gives a very practical way to compute if Homotopy groups are equivalent. This is used in Manifold theory all the time.

12.4. Cohomology

We will motivate the study of Cohomology, which is worth a full course in its own right.

Recall that if V is a vector space over a field, then the dual vector space is $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$, which has the same rank, but there is no natural isomorphism of V and V^* unless a basis is chosen.

Note that if $V \xrightarrow{f} W$ represented by matrix A, then $V^* \xleftarrow{f^*} W^*$ is represented by A^t.

Now suppose we are using coefficients in a field $\mathbb{F}$. We can apply dualization everywhere:

Before we talked about a chain complex

$$C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_1} C_0,$$

now we get the *co-chain complex*

$$C_i^* \xleftarrow{\delta^i} C_{i-1}^* \xleftarrow{\delta^{i-1}} \dots \xleftarrow{\delta^1} C_0^*,$$

where we normally let $\delta^i = \partial_i^*$. Then we get

$$(H_i(X))^* = H^i(X) = \frac{\mathrm{Ker}\,\delta^{i+1}}{\mathrm{Im}\,\delta^i}.$$

Then we get all the same axioms for $H^*(X)$ except with arrows reversed. For example, for $X \xrightarrow{f} Y$ the induced map is $H^k(X) \xleftarrow{f^k} H^k(Y)$.

So how is this different from Homology, and why bother?

In many settings, eg. Analysis, functions on areas are more the focus than the areas. For example, if we took a loop, $C_k(X) = \bigoplus_{k\text{-cells in } X} \mathbb{Z}$, then

$$C^k = \mathrm{Hom}(C_k(X), \mathbb{Z}) = \mathrm{Hom}\left(\bigoplus \mathbb{Z}, \mathbb{Z}\right) = \bigoplus \mathbb{Z}.$$

Another reason is that there is a bilinear map $H^k(X) \times H^\ell(X) \xrightarrow{\cup} H^{k+\ell}(X)$, called the *cup product*.

However, it is anti-commutative, in the sense that $\alpha \cup \beta = (-1)^{\dim\alpha \dim\beta} \beta \cup \alpha$. The idea is that $[\alpha] \in H^k(X)$ means α is a linear function on $C_k(X)$, so it is

specified by its value on a basis of k-simplices of X. So for a k-simplex $\Delta^k \xrightarrow{f} X$, we have $\alpha([f]) \in R$. Given $\alpha \in C^k(X), \beta \in C^\ell(X)$, and $f : \Delta^{k+\ell} \to X$, we define $\alpha \cup \beta(\Delta^{k+\ell} = \langle v_0, v_1, \ldots, v_{k+\ell}\rangle) = \alpha(\langle v_0, \ldots, v_k\rangle)\beta(\langle v_{k+1}, \ldots, v_{k+\ell})$. We need to check that it is well-defined on $H^*(X)$. Well $\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta \pm \alpha \cup (\delta\beta)$. We have $1 \in H^0(X)$ where $1(\mathrm{pt}) = 1$. Then $[1] \cup [\alpha] = [\alpha]$.

So in this way Cohomology gets an algebra structure.

An interesting example to work out is the following: Let $X = S^1 \vee S^1 \vee S^2$, and $Y = S^1 \times S^1$, then they have the same $H_*(X), H_*(Y)$, so additively the same $H^*(X), H^*(Y)$, but different multiplicatively. So additively, we have generators $1 \in H^0(X)$, $\alpha, \beta \in H^1(X)$, $\gamma \in H^2(X)$, and we have a similar story additively in Y. However, multiplicatively $\alpha \cup \beta = 0$ in $H^2(X)$, but in Y, $\alpha \cup \beta = \gamma$.

So we can see that these spaces are not homotopic by noting that their Cohomology groups are different multiplicatively.

The big fact is that if we have $X \xrightarrow{f} Y$, then $H^*(X) \xleftarrow{f^*} H^*(Y)$ is not just linear, but a ring homomorphism compatible with the multiplication:

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta).$$

Now note that by our definition of a co-chain, we have an evaluation of co-chain on chain $\langle \cdot, \cdot \rangle \in R$. Then we have $\langle \delta\alpha, A\rangle = \langle \alpha, \partial A\rangle$. In other words, $\int_A \delta\alpha = \int_{\partial A} \alpha$, which is Stokes Theorem. So there is a formulation of Cohomology in the langauge of differentials, which is called De Rham Cohomology, and then we can get the same isomorphism of Cohomology groups, where $dx \wedge dy \leftrightarrow \alpha \cup \beta$.

The following is an interesting example of a non-zero product:

Example 12.4.1. In $\mathbb{R}\mathrm{P}^n$ we have $H_*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}$ for $0 \leqslant * \leqslant n$. However $H^*(\mathbb{R}\mathrm{P}^n, \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1} = 0)$, where x^k is the additive generator in dimension k. On a manifold this multiplication in $H^*(\cdot)$ is dual to what is called the *intersection product* on $H_*(\cdot)$.

Multiplication in cohomology determines intersections in homology, but this only works on manifolds.

12.5. Eilenberg Obstruction Theory

This theory answers the following problem:

The extension problem in Topology is the following: we have for $A \underset{i}{\hookrightarrow} X$ a map $A \xrightarrow{f} Y$. The question is if there is a map $X \xrightarrow{g} Y$, called an extension, so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, hook, "i"'] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

Figure 12.5.1.

The answer is clearly not always, since if we took $S^k \hookrightarrow D^{k+1}$. Then there exists an extension g only if $[f] \in \pi_k(Y)$ is 0.

So is there a complete answer to when we can extend a map? Well Eilenberg Obstruction Theory measures the obstruction to extending a map in stages.

Remark 12.5.2. A basic principle is the Homotopy Extension Principle, which says that this depends only on the Homotopy class of f.

In other words, the problem depends on f, but in fact it is the same for maps homotopic to f. Now this is not immediately obvious.

To prove this on cell-complexes, say $f_1 \underset{h}{\sim} f_2$ as maps $A \to Y$, we want to show if f_1 extends to X then so does f_2. The proof will be by induction on the cells of X not in A. Now think of $X = A \cup \text{cells} \cup \ldots$. So for adding one cell e^k, we just need the following key lemma, since all we care about is what is happening on the edge of the cell.

So we have a homotopy from $e^k \times 0$ to $e^k \times 1$, and we have an extension of the boundary map f_1 to the whole disk $e^k \times 0$. We want to know if we can extend f_2 the same way. But we have a retract r that takes the "cylinder" $e^k \times I$ to the "sides" and the "bottom". So we can just compose.

This tells us that the extension problem depends only on the homotopy class of f.

So the Eilenberg obstructions compute whether this can be done inductively on $A \cup (k\text{-skeletons of } X)$. We will hit problems: the trouble is that when we try to extend the map, for each cell, we will encounter an obstruction given by the homotopy class of the map on its boundary.

So this tells us that for each relative k-cell (X, A), we get an element $\pi_{k-1}(Y)$. So on k-cells we get a homotopy class. So this altogether actually defines an element of the co-chain $C^k(X, A; \pi_{k-1}(Y))$. This has co-boundary $\delta = 0$. So we end up in $H^k(X, A; \pi_{k-1}(A))$.

The problem is that as we extend the map, we make choices, and what we get in the obstructions could depend on the choices we make. So while extending we might get an obstruction that reflects a bad choice 50 dimensions ago, for example.

So in practice Eilenberg Obstruction Theory is useless except in very special cases.

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